JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

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JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Opening Address by the President, S. G. Warner, Esq.

[Delivered 4 December 1917.]

Gentlemen,—We meet to-night to open another War Session. The hopes, probably entertained by many of us, that the twelve months which began for us at our similar gathering of November 1916, would see the end of the struggle, have not been fulfilled. Still our main activities are suspended, our numbers depleted, our anxieties prolonged, under the shadow which overhangs the world. We are at all events one year nearer to our goal, as a people; but how distant its attainment still may be we are as unable as ever to say. At least, however, this may be asserted: that we, in common with our fellow countrymen, our kindred oversea, and our Allies, are as resolute as ever that attained that goal shall be: that we, in common also with them, are prepared determinedly to endure such privation, suffering or sacrifice as may be necessary to that end; resolved, in the words of our great poet and patriot of a past day, that we will not

"bate a jot
"Of heart or hope, but still bear up and steer
"Right onward."

You will be interested to hear the numbers, brought down to the present date, of members and probationers of the Institute who have joined His Majesty's forces on land or sea; the total is now 390, and of these 44 have fallen. Since the last announcement of this kind, at our Annual Meeting in June, ten more names are included in that Roll of Honour, devoted to the memory of those whom we shall see no more.

They have laid down their lives for their country, and no comment could do other than weaken the force of that simple statement of fact. In each case, on your behalf, an expression of sympathy has been sent to those from whose homes they have gone. They have left us, but they are not forgotten.

Two other losses by death have to be recorded. Mr. Andrew Hugh Turnbull, who died last month, had been a Fellow of the Institute for nearly forty years. His name is naturally associated with the great Life Assurance Office in the service of which so much of his life was spent and to the success of which his management so substantially contributed. For four years he served on our Council; he was a member of the joint Mortality Investigation Committee; for two years he held the office of President of the Faculty of Actuaries; and he was the author of two useful books of Actuarial Tables. Those who knew him intimately bear witness to the wisdom of his counsel and the charm of his character. As Fellow of both Institute and Faculty he rendered good service to each, and by each his name will be held in honoured remembrance. The death of Mr. Edward Algernon Newton, who became a Fellow in 1865, leaves on our Roll only four names which have a longer record in that respect. As he died in his ninetieth year, it follows that his active business career belonged to rather a bygone day: and as throughout his retirement he had few opportunities of meeting us he is probably to many of us little more than a name; but his work was well and faithfully done, and in losing him we break one of the few remaining links with a distant past.

Among the events of the war during the past twelve months there is one of outstanding importance which has an aspect of special interest to British Actuaries. I refer to the entry into the contest, on that which we hold to be the side of righteousness and freedom, of the United States of America. I suppose there is no one of our race who did not feel a stirring of the blood and an uplifting of the spirit at this act of the one great English-speaking nation outside the British Empire. Beyond this, however, we have our own keen interest in and warm friendship for our brother actuaries across the Atlantic. The Actuarial Society of America has been long and honourably known to us for its great services to our science, and for the freshness, originality and ability with which it has handled many actuarial problems. In these

circumstances there occurred during the summer an interchange of messages, which are doubtless familiar to you since their publication in our Insurance Press, but which I venture to reproduce here so as to secure for them permanent record in the pages of our *Journal*. The first was as follows:

"The members of the Actuaries' Club, London, send "greetings to the members of the Actuarial Society of America. "They hail with joy the entrance of your great country into "the struggle for justice and freedom among the nations."

This elicited the following reply:

"The Actuarial Society of America in annual meeting "assembled acknowledges with deep appreciation the message "of friendship from the Actuaries' Club, and unanimously "sends most cordial greetings to the Actuaries' Club and to "the members of the Institute of Actuaries. American and "Canadian members have worked side by side in this Society "for many years, and we all now rejoice that America and "the British Empire will hereafter stand side by side in the "great struggle for freedom and civilization."

It will be observed that the American message includes in its scope not only the senders of that to which it is a response, but our Institute as a body. It seems appropriate therefore that on this, our first opportunity since its receipt, we should send a reply reciprocating the feelings expressed. It is my intention accordingly, before our proceedings close, to propose for your acceptance a resolution which will answer that purpose.*

On the last two occasions of an inaugural address in this Hall, the war and its problems naturally formed the central theme. To-day the situation, developed during the last twelve months along broad lines of strategy and statemanship, hardly seems to present material making it necessary for us as a professional body to devote another entire evening to its consideration. The great conflict has had its chances and changes, its encouragements and disappointments. Steadfast effort and endurance are more than ever necessary, but perseveringly continued they will not fail of their reward. The financial burdens of the nations are heavier, but there is no weakening

* [The resolution, which was unanimously adopted at the close of the proceedings, was in the following terms: "The Institute of Actuaries in General Meeting assembled, acknowledges the greeting sent through the Actuaries' Club by the Actuarial Society of America, cordially reciprocates the friendship that inspired it and rejoices in the fellowship, as comrades in arms, of the British and American peoples."—Eds. *J.I.A.*].

of the will to bear them rather than, on any excuse, to throw the fruits of past sacrifice away. We have all been hard at work, under the double strain of depleted help and the new problems which constantly arise. As instances of the latter, some have had to undertake a study of the intricacies of new foreign and colonial Income Tax Acts, which is in itself a liberal education; others have been facing questions of pensions to combatants and dependants which are increasingly complicated and difficult; while the help of some of our number has been sought, and effectively given, in the arduous statistical labour necessary in connection with our shipping. The national finance involves issues exceedingly complex, and sometimes keenly controversial; but to-night, when controversy is a thing to be avoided, it is well to be assured that all actuaries are united as regards the one supreme financial duty, the continual enforcement upon our people of the need for an unflinching and untiring economy. The voluntary expert help which members of our Institute have in such large measure given to War Savings Committees, and to important Funds created by and connected with the war, we record not with any claim to have done more than patriotic citizens' duty, but with satisfaction that the call to such duty has not failed to find a willing response.

We learn with great satisfaction that Mr. T. B. Macaulay, a Fellow of this Institute by examination, has been appointed Chairman of the National Committee of Food Resources in Canada. We congratulate him on his appointment, and the Canadian Government on its selection; and record the event as an honour to our body and our profession.

Leaving now this battle-shaken ground of the present, I invite you to follow me in a line of thought more directly concerned with our own profession. It may seem to lead us far away from the practical actualities of our time, but in the end we may find that it is not without real bearing on the great questions which are likely to be our close companions in a not very distant future.

When we last gathered together in this Hall, in June of the present year, it was to take part in the seventieth Annual General Meeting of the Institute of Actuaries. We belong therefore to a body which has completed the term of three-score years and ten. I hope we may say that its eye is not dim, nor its natural force abated. Yet the attainment of such

an age may perhaps not unfitly be regarded as marking a kind of epoch; especially when it happens to coincide with a world convulsion that seems likely to lead to a social reconstruction of a character which we can just now only dimly surmize. We have some excuse for looking back into the past, an exercise which may give us some help in forecasting the future, and which is at all events a much surer and I think too a more profitable process.

In November, 1853, just over sixty-four years ago, a paper was read to the Institute by Mr. H. W. Porter "On some points connected with the education of an Actuary." The ideal there sketched by the author is a high one. He stipulates for "a good general education—such a liberal education as a university course provides." This is to include "a good useful knowledge" of Latin and Greek. These languages are, among other uses, to assist him should his Directors require him "to prepare a nosological table of the causes of the deaths that have occurred in the office." In the same connection he is to have "some knowledge of the physiology of the human frame." Modern languages are also recommended, and at least an acquaintance with French. A scientific knowledge of statistics is required, and a study of schemes of social improvement as affecting longevity. Enough law to become "a valuable coadjutor of the legal adviser of his Company" is thrown in. He should also be "a good accountant", "a ready correspondent", and "well read in all the literature appertaining to his profession." A knowledge of finance is also essential. All this is added of course to a thorough masterly knowledge of arithmetic and algebra and of the science of life contingencies; with a warning that geometry should not be neglected. "A perfect actuary", says Mr. Porter, "should be a kind of admirable Crichton." There seems indeed to be little of the admirable Crichton left out, except athletic accomplishment: and this might have been included, had golf been a national pastime in 1853.

To criticize fairly such an utterance as this we must try to place ourselves, in imagination, back at the date of its delivery. The Institute had been in existence for six years only. Examinations were in their infancy. Text-books, as we now understand them, were unknown. The outstanding feature of the paper is its obvious motive of impressing on

the intending actuary a sense of the honour and dignity of his profession. There could be no worthier object. It is as true now as it was then, and as it will always remain, that in proportion as that sense is felt our work will be done well. From the nature of the case such an essay, when it was written, partook largely of the nature of a forecast. The great critic and illuminator of all forecast is experience. Sixty-four years of development lie between us and this one. It would be strange if by the aid of their light we were not able to-day to revise some of its conclusions; to place its main motive in a setting more adapted to the conditions and demands of to-day.

What chiefly occurs to us, then, when we make this attempt, is that while the author's purpose is finely conceived, there is something lacking. To say that the perfect actuary should be an admirable Crichton is not, after all, to say very much. In a sense it is true; but in that sense it might as truly be said of the perfect banker, the perfect accountant, the perfect auctioneer. In the comprehensiveness of the ideal all distinctiveness is lost; and distinctiveness is vital. Distinctiveness then we must find. It is not to be expressed in a sentence, or a formula; but I think it will grow upon us with sufficient precision by a consideration, even in the brief and inadequate fashion which alone is possible this evening, of some features of the history of our science; while at the same time the consciousness of its dignity and honour should be intensified.

The beginning of the story, familiar enough to most of us, takes us back far beyond the existence of the Institute, to the middle of the seventeenth century; when Latin was still the "Lingua franca" for learned men of all nations, when such men were still comparatively few in number, and when fresh ideas, once formulated, spread with a ready freemasonry amongst them and were eagerly received and developed. There is no intention, as there is no need, to retrace in detail ground familiar here to us all, but it is surely something to be proud of that the germs of the investigations which made our science possible are to be traced to the brain of a thinker of such world-wide eminence as Blaise Pascal. The possibilities latent in the application of the principles of probability to questions dependent on the duration of human life were readily perceived and diligently pursued by the

mathematicians of northern Europe; England was not found wanting, and we find ourselves in the line of succession with the great name of Edward Halley. Thus set flowing, the current of the new thought gradually set towards and finally settled in our own country. There the speculations of the philosophers consolidated themselves into practical form. This was natural, for the British intellect is on the whole rather practical than speculative; but it furnishes another reason why we, as Britons and actuaries, should honour our craft.

It is possible that a purist might take objection to the words "actuarial science", pointing out that what is meant is not a science, but a particular form of applied mathematics. Whether this be strictly so or not it will be convenient for our purpose to use, for conciseness' sake, the word rather than the phrase, as there can be no misunderstanding what is meant.

Taking this liberty, then, if I may, it is to be observed that a new science, born into a world of active thought, goes through, in the earlier period of its development, what may be called a romantic stage. It attracts attention from, and excites interest in, a wide spread circle of thinkers and investigators. This is a historical aspect of actuarial research which we perhaps do not think of very often. In these strenuous days especially we have little leisure for contemplating the past. I do not know that I can plead in defence of such an exercise that it is likely to serve any immediate practical end, though indirectly perhaps it may. But anyhow, in the attainment of our seventieth birthday we are entitled to spend a few minutes on looking back at our past; and at the earlier years, before our corporate existence began, when the forces which led up to our existence were shaping themselves in the life of the country.

To begin with, is it not significant that Dr. Richard Price, that redoubtable figure of the later eighteenth century, classical scholar, Unitarian preacher, Radical politician, and adviser (with questionable judgment) of William Pitt on the National Debt, should have found time among all his other affairs to construct the Northampton Table of Mortality? That is typical of the rising flood of interest surrounding the childhood of the new science. But there are other names more immediately appropriate to our purpose. One thinks

for instance of George Barrett, the Surrey farmer's son, who in the scanty leisure afforded by hard daily work fought his way unhelped through the mysteries of mathematics; toiled during twenty-five years, while by other labour supporting himself and dependent relatives, at the compilation of a mass of assurance and annuity tables, and arrived by independent discovery at the Commutation Method. In obscurity and isolation, his only help for the arithmetical work required being that of his home circle, he elaborated the invention which was to prove so powerful an instrument in the actuary's hands. He got the position of actuary to an assurance company, but held it for only two years. His failure to obtain public recognition for his work shortened his days. It seemed, he said, as if "thou shalt not prosper" was written on all his undertakings. One wishes there could have been granted him a prophetic glimpse of what his invention would mean to those who followed him.

A considerable figure of a very different kind, who also for a time held the post of actuary to an assurance company, was Charles Babbage, who combined mathematical genius with a good deal of rather ungovernable eccentricity. He was a man of wit, for when he along with some Cambridge associates was trying to get the notation of Leibnitz for the Calculus established in place of Newton's he described himself and his friends as "promoting the principles of pure *D*-ism in opposition to the *Dot*-age of the University." He had a European reputation, and has left to actuaries one conspicuous gift in his well known Table of Logarithms. His versatile intellect ranged over many fields. We know his little book of 1826 on the life assurance offices of that day. Even now it may be read with profit. But his mental activity was endless. Co-operative workshops, the interpretation of ciphers, the prevention of railway accidents, systems of signalling by lights, the construction of automatic chess players, are only a few of the subjects which engaged his attention. But his ruling passion was the art of mechanical calculation, for which he had the honour of receiving the first Gold Medal awarded by the Royal Astronomical Society, and in which he deserves from all actuaries the tribute due to a pioneer. His devotion to it forms a pathetic story, for it was there that what he deemed lack of proper recognition made him a permanently embittered and disappointed man. At

South Kensington Museum are still to be seen parts of the giant machine on which, out of his own fortune and the public money, some £23,000 were spent; and although it remains there, in a sense a failure, we who enjoy the benefit of later developments, and may yet see more powerful developments still, may not unfitly recall the words "other men laboured and ye are entered in to their labours."

Considering these cases, we are led to what seems to me one of the most remarkable and attractive features of this early period of our science: the number of men of first rate ability, not interested in it primarily for any professional or commercial reason, who were drawn to it, and worked at it, as a new mathematical development full of powerful possibilities. Francis Baily, for instance, we know actuarially as the author of a treatise on Annuities and Assurances which was for long the standard work on the subject. But Baily was far more than this. A successful stockbroker and exceptionally able man of business, his outstanding reputation was won in astronomy. A double gold medallist of the Royal Astronomical Society, one of its fourteen original members, and at his death its President, his labour on star-catalogues, and his success, after five years' arduous labour, with the "Cavendish experiment" for measuring the earth's density, give him a monumental place in astronomical research. He was conspicuous in trying to obtain for George Barrett that recognition of his work by the Royal Society, the failure to receive which was one of that luckless man's heaviest disappointments. Baily died three years before our Institute was founded, but all actuaries who can appreciate a single-minded devotion to the pursuit of knowledge must feel proud of the association of such a name with our science.

The mention of Francis Baily's name suggests another, and I venture to think a greater one, that of his intimate friend for many years, Augustus De Morgan. It is probably a matter of personal taste and temperament, but I confess that to me De Morgan has always seemed one of the outstandingly attractive figures of the Victorian age. His versatility, his wit and humour, his simplicity of character and inflexible devotion to principle, combined with his strength and acuteness of intellect and the width and variety of his learning, constitute a combination which never fails to fascinate me. He comes closer too to our own day. All

his best characteristics are shown in his inimitable "Budget of Paradoxes", which I think it one of our greatest honours to possess in serial form in our *Journal*. As for his humour, it is of the kind that makes the whole world kin. Who could feel anything but kindness for a man who having in a letter to a friend emphatically delivered his mind on a serious subject, instead of winding up with the familiar Latin tag "Dixi", writes "Richard's optic"? Further, as well as being an outstanding man he belongs to a remarkable period: a period typified by that Association for which so much of his best work was done, and which in its twenty years' existence accomplished so much good, "The Society for the Diffusion of Useful Knowledge." That time was a remarkable one in our intellectual history. The spirit of new knowledge was in the air. It pointed out possibilities in many directions, and gave birth to many enthusiasms. Some of them are by this time chilled and forgotten, but we are none the better for that as a people, and some of us envy the veterans, now so few surviving, who can look back on that period in the spirit of Wordsworth's great phrase "Bliss was it in that dawn to be alive." For along with this joy in the new learning went a limitless faith in the enterprise of imparting it to the people at a cheap price: so we had the fruitful labours of Charles Knight, the Chambers Brothers, and John Cassell. The Society I have mentioned did good work for our own subject on this path. Through its press issued our familiar friend "Jones on Annuities", De Morgan's great treatise on the Calculus, and, better than all, the "Penny Cyclopædia", containing several actuarial articles by him among the 711 contributions which witness to his exhaustless energy. Our own *Journal* also is enriched by similar work from his pen, and we have his separate treatise on Probability as applied to Life Contingencies.

Two associated names of remarkable men who have contributed to our science, if we may so say, from without, are those of Peter Gray and William Orchard. Gray had a long life, Orchard died in youth. Separately and in concert they are responsible for some of our most useful books of reference. Gray's application of the Gaussian logarithms to continuous table construction was a triumph of ingenuity; while Orchard's development of a fundamental relationship of two functions into conversion tables produced a work

which became perhaps more intimately the actuary's daily companion than any other.

Peter Gray was an Aberdonian. In mentioning his name we, for the first time on our list, "crossed the border." I would do so yet again. No actuary who honours his profession should forget Edward Sang. Fond of the study from boyhood, Dr. Sang developed early into a brilliant and versatile mathematician. Engineering was his original profession, but the greater part of his life was spent as a teacher of mathematics in Edinburgh. His work in mechanics, the science of naval architecture, hydrostatics, and astronomy was original and valuable; while the story of how he impressed the population of Constantinople by the prediction of a solar eclipse reads like the famous chapter describing a similar exploit of Allan Quatermain in Rider Haggard's well-known novel. He was responsible for a table of Logarithms very extensively used by our profession; while another, to 15 places, in 47 volumes, the result of forty years' labour, lies in state in the rooms of the Royal Society of Edinburgh, a monument of scientific devotion. He too felt the attraction of actuarial science, compiled extensive annuity tables, wrote essays on life assurance, and a treatise for students on life contingencies, which, notwithstanding a rather unfamiliar notation and a severely compressed style, is of great value.

I close this catalogue, which I trust has not become wearisome, with the name of Dr. William Farr. To him, as his biographer fitly says, the science of vital statistics may be said to owe its foundation. He is interesting as contributing a new element to actuarial research. He approached the subject as a medical man and a statistician. To him we owe the origin of that co-operation of the population registrar and the actuary, which for that great branch of actuarial science which is concerned with national vitality we may call the breath of life. In his hands the Registrar-General's report became instinct with scientific interest, and he created the English Life Tables. Without such work, enterprises in the interest of national health and well-being, which have done much and may yet do much more, would have been impossible.

Now I have not brought these names and reminiscences together for the purpose of giving new information, for I

suppose little, if anything, has been said which is not familiar. The object is of another kind. Here we have a number of eminent men, varying widely in experience, training and surroundings. They are all attracted from their individual paths of mental activity to study something which lies outside them, and that the same thing. When we view the subject thus historically, we can be in no doubt about what that thing is. In the application of the science of probabilities to the facts relating to human mortality, the statistical ascertainment of those facts and the application to them of that science, combined when the purpose was financial with the laws of compound interest, they found something new and absorbing. Here then we have arrived at the distinctiveness we sought, and can thus, while doing I trust full justice to the spirit of Porter's ideal, make its comparative formlessness symmetrical by giving due place to its centre of gravity. For in that same study which in the days of its youth drew to itself the attention and the help of these distinguished workers in other fields, and the possibilities of which are unexhausted, is, and always must be, the essence and the distinguishing mark of the actuary's vocation. As regards the accessories indicated in that utterance of 1853, it would be idle to deny their value: but as accessories, if we would get the true perspective of our subject, they must be considered. As knowledge on all sides multiplies, and specialization is more and more forced on us as a necessity, it will be for individuals to select from these subsidiary paths as temperament and circumstances dictate; but the philosophy of the matter so far as they are concerned will be best expressed by the spirit of Dr. Johnson's famous saying about Greek: "Greek, sir, is like lace. A man gets as much of it as he can."

I have spoken of the romantic stage of actuarial science. There is one chapter of its history, very different from the rest, to which that description seems to me to apply in some ways more deservedly than to any other; and which no attempt at retrospect, however inadequate, can afford to leave unnoticed. The early years of the nineteenth century saw the beginning of an attempt among the English working class to stimulate, and organize into definite form, the Friendly Society movement, which in scattered and spasmodic fashion had existed for more than a hundred years. The

story of its subsequent development to its present stage cannot of course on an occasion like this be sketched in the merest outline, even were the knowledge and the power for such a task mine, which they are not. All I should like to attempt is a very brief mention of how actuarial principles made their way into Friendly Society finance. We are fortunately able to trace this from the full records we possess of the rise and progress of one great typical organization, the Manchester Unity of Oddfellows, through which that work was done. The story is told in Mr. R. W. Moffrey's most interesting book "A Century of Oddfellowship", to which I gratefully acknowledge my indebtedness. During the first half of the last century, while energetic leaders were successfully enlarging the organization, and building it up into a coherent whole, the more farsighted among them were made more and more keenly alive to the fact that among the units which they were thus binding together there was, and most naturally, a profound ignorance of the elements of the science of adjusting contributions to benefits, an observance of which was essential to avert disaster. The occasion was grave, and was met with courage. We can form no adequate idea of the difficulties of suspicion, inertia, and disbelief, which from the nature of the case would lie in the path of the reformers. The leader for the emergency was found in Henry Ratcliffe, appointed corresponding secretary of the Unity in 1848. A self educated man, he grasped the necessity of the situation and devoted his life to its demands. In 1850 he published his famous "Observations on the Rate of Mortality and Sickness existing among Friendly Societies", with appropriate assurance and annuity tables. We do no injustice to the previous labours of Finlaison and Neison in saying that this work, in all the circumstances, was epoch-making. At the mass of material there presented he had laboured long, for from fifteen to seventeen hours a day. His salary for the whole duties of the Secretaryship was £150 per annum, and for the extra work involved he received a personal remuneration of £50. He lived to repeat this investigation twice, in 1862 and 1872, each time in the light of later experience. His personality and energy had won their battle. On his death in 1877, while his full ideal had not been attained, he had seen and done enough to feel well assured that the light would spread till throughout the

great organization for which he had lived actuarial solvency and security should prevail. That great possibility was due to his own untiring labour. Mr. Moffrey well says it justly entitles him "to be remembered as the father of friendly society actuarial science." His work was taken up by Reuben Watson, long his close colleague in the effort, and himself also a self-educated actuary. He too has passed away, but the influence of the achievement of these men and their like has saved one of the greatest developments of national thrift from failure. When we look back as actuaries on the development of our science, let us pay them the honour that is their due.

There is an interesting reference in Mr. Porter's essay to the problems which come before the actuaries of Reversionary Investment Companies, which, he says, "are of a far higher order than the ordinary Assurance Office calculations." What was probably meant was that such questions presented greater variety, called for more individual initiative, and were less amenable to treatment by stereotyped formulas. The subject is an interesting one, and contributes a feature of its own to the history of actuarial development. This was certainly a case in which our science opened up a distinct path of public service. It is difficult to form any definite conception of what happened to reversionary securities before scientific ways of dealing with them began to be established. There are no accessible materials available for such a purpose. We can only arrive at a general suspicion that transactions of this kind must have been carried on in obscurity, that they were probably associated with speculative dealings of rather a rapacious and discreditable kind, and that the atmosphere of the whole business was unattractive. When actuarial science showed how to attach to such prospective interests fair and legitimate present values, the air was cleared. Securities of this character took their proper place in the financial market, as proper material for straightforward bargains in which buyer and seller, or borrower and lender, might deal on equal terms. So far as the actuary himself was concerned, the effect was educational in a valuable way. He was brought into relation with a widely varying body of investments, which he had to value in a way that involved forming an estimate, not of their present worth but of that which they might be expected to possess in a more or less distant future.

This meant a forecast widely different from those inherent in his other work. The probabilities of sickness and death in the case of a Stock Exchange security are a good deal more complex, and difficult of appraisement, than those of human life. To deal with them needed knowledge, both actual and historical, of the money market, and the judgment which from the past and present could make some practical estimate of the future. Gradually, as technical methods increased in power and exactitude, contingencies of the most intricate kind were brought within the scope of ready calculation, and a large and successful business has been developed. Actuarial science may justly be credited with the achievement of having purified and organized on an equitable basis a branch of finance which, by releasing for present use capital not otherwise immediately accessible, has doubtless often played a useful part in the development of national resources. The conventional notion that it is the spendthrift only who raises money on his expectations is as frequently wrong as most conventional notions are. It may just as easily be the man who can put immediate capital to judicious use, and to help him is, as I have said, a public service.

Reversionary transactions, in early days, before we had Institute or *Journal*, must have been the material for many ingenious devices, and elegant approximations, among the older actuaries, of which no record remains. That reflection suggests a much larger one, about the whole growth of our science during that now distant time. When we consider what scanty materials, compared with those we now possess, were in the hands of our predecessors, we realize more fully what considerable men they must have been. The conditions were favourable to the development of individual capacity, independent views, original thinking. Before the Carlisle Table saw the light, or such a thing as an Offices' Experience Table was thought of, some of the older actuaries worked on Tables of their own construction from their own records. Their wisdom was contained in note books long vanished, conversations long forgotten. "There were great men before Agamemnon", but all these memorials are faded and fled. We shall not recover them. We possess some of their pamphlets (notably William Morgan's addresses to the Equitable Society), and some of their Tables; but the setting of the whole, the personal atmosphere that took the place of

our present organized publicity, all that, which was of the essence of the matter, is gone beyond recall, with the entire social order of which it formed a part.

When the romantic formative period of a science is over, it gradually settles down into one of ordered systematic progress, which while supremely useful has less about it to attract the imagination. I suppose, broadly speaking, we may say that the original Actuaries' Club represented the old order, and the Institute the new. That I know is by no means a perfect division, but only a very rough one. We possess, of course, on the Institute roll the names of many illustrious men, active in its inception and throughout its earlier years, Finlaison (its first President), Jellicoe, Brown, and others, who were links between the old world and the modern day. Even down to our very recent history, we have had with us the honoured presence of some whose recollections and whose spirit united the present and the past: one in particular, Ralph Price Hardy, who seemed to carry with him a charm from the days of the fathers of the profession. It was evident in the predominance of his personality over his formal written contributions to our proceedings—substantial as their value was; in the seeds of suggestiveness and inspiration sown by him in the minds of the younger men around him, and bearing fruit in their lives and work. Gradually, however, the influences which had shaped the beginnings of our progress consolidated themselves into a middle period with characteristics of a different kind. The time of the pioneers was over. First principles were established, and the lines of further advance laid down. The attention attracted from distinguished mathematicians outside the ranks of the profession inevitably slackened; and the economic conditions of the time associated its members more and more closely in one practical interest, the propagation and building up of the business of life assurance throughout the nation. This atmosphere is evident, for instance, throughout the whole of Mr. Porter's essay, and obviously controls his ideal. It would be the greatest possible mistake, however, to suppose that this development coincided with any diminution of powerful mental activity. The intensive work showed certainly no sign of such a tendency, but was carried on with keenness and vigour. Looking back upon that time, and endeavouring to view it in its true perspective, we see running through it as

its central outstanding feature what we should naturally expect; the endeavour to perfect the statistical basis on which the work was to be founded. If the actuary's main sphere for the period was to be that of life assurance, the existing tables of mortality, constructed from general population returns, were not the most suitable. The tools must be more closely fitted to the task, and so we have the great repeated efforts of the British actuaries to obtain tables representing the experience of assured lives. It has been a long-continued and persevering labour, taking form in three successive bodies of results; each in volume of content and elaboration of treatment an advance on its predecessor. Into it a mass of the best ability of our profession has gone, while the attendant processes of graduation, and of appraising the influence of medical selection, have called forth mathematical skill of a high order. When we were dealing with the farther past, it was not very difficult to select for mention a few outstanding names. As the time approaches nearer to our own, the number increases; and it is not easy, without some omission, to make a choice. I can only ask forgiveness if I am guilty of that, but I do not think there will be any dissent when I recall as honourably associated with the great enterprise just described in one or other of its branches, the names of J. A. Higham, William Sutton, W. M. Makeham, James Meikle, T. G. Ackland, W. S. B. Woolhouse, and George Francis Hardy, who have passed away, and, still with us, Dr. T. B. Sprague and Mr. George King.

Grouped around this central undertaking, the period was fertile in much other research. The quest for a law of mortality, leading to the brilliant generalization of Gompertz, which with its modifying development by Makeham has proved so powerful an actuarial instrument, one would rank next in importance. It is rather curious that Mr. Porter's theory of actuarial education lays comparatively little stress on the higher mathematics. The history of our science in its subsequent development has been distinctly to emphasize their value. It is not uninteresting to trace the process. Actuarial study could not be long pursued without recognition of the fact that it was primarily concerned with a branch of the doctrine of series: and further that from the nature of the case it was proper to assume continuity. This led directly to the Differential and Integral Calculus. For some time

there was hesitation about including an elementary knowledge of these in the examination syllabus. Ingenious demonstrations of formulas based on assumptions of uniform annual distribution of deaths were the result; but the cumbersomeness of these was evident, and an epochal step was that of Woolhouse, when in his essays on the "Continuous Method" he boldly applied the Calculus both to mortality and interest, thus obtaining expressions far superior in elegance and simplicity and more closely in accord with the real conditions. The Gompertz-Makeham formula already mentioned, with its reliance on transcendental functions, further emphasized the value of a free use of methods in advance of those of ordinary algebra. On all hands and in unexpected ways this value revealed itself. The brilliant use of summation formulas in dealing with complicated survivorship contingencies, which had previously eluded any expeditious and exact method of calculation, may be given as an instance. The cumulative evidence from all sides of the power and benefit thus derived could not be resisted; and, first optionally, then compulsorily, the elements of the calculus were introduced into the examination tests required of our students. No such experiment has ever been better vindicated by the result. The intending actuary of our day is in the position, enviable by some of us whose training belongs to an earlier period, of being, at a time when the mind is plastic to new impressions, familiarized with the use of a mathematical tool of first-rate power, the mastery of which at a later age is very much more difficult to attain.

If it is a correct generalization to say that there have thus been two periods in British actuarial history, distinguished from each other by some recognizable characteristics, the same reasoning will I think lead to the conclusion that there is yet a third, of which we may say that it is still opening before us and that we as yet stand on its threshold. When looking at the subject in this light, we are not to expect any sharp dividing lines. The distinctive features of change merge into one another. Many of those which become evident have long before been latent, and it is only when a general view over a sufficiently wide reach of time is taken that the differences are seen. The principle at work is really the reaction of the material to the conditions surrounding it. It would, for instance, be a mistake to suppose that none of

the earlier actuaries were conversant with such ideas as those of framing a Life Table from assurance experience or analyzing the effect of selection. The growth and consolidation of life assurance business made such problems, as years advanced, both practical and practicable.

So in the development to which I now refer, we have to think not of a transfer of the actuary's activity from one sphere to another, but of the opening to that activity of a wider field, while preserving that of the past; and, in that process, of bringing into concrete, systematized form, much that has long existed in a more or less fragmentary and fluid state. A good instance may be found in the work so admirably done for Pension Fund Tables by the late Mr. Manly. Such Funds had existed, and been subject to actuarial advice, long before. It was when the subject came to occupy wider public attention; when consequently the demand for guidance upon it, in the varied and complex forms it was assuming, grew more insistent, that its codification took material shape. Now this is but one minor case of a general tendency, of which I feel assured we have as yet seen only the beginning. We may measure its effect in another and quite different way. Speaking to us in 1881, one of our most experienced Presidents could say:—"Men have occasionally attempted to "practice as pure actuaries, and have put a brass plate on "the door; but whether they have ever taken sufficient fees "to pay the rent of their offices is doubtful." We need no more striking evidence of what is taking place than to compare those words with our experience to-day. There is the widening field, and it is one of general public service.

Let me, however, emphasize what was said just now, that this is a widening, not a transfer, of activity. The actuary's place in the life assurance sphere is as central, as essential, as ever. Much as the labours of the past have clarified and codified, individual judgment and initiative will always be indispensable. As the mass of observed experience increases, fresh revision of it for tabular purposes will be necessary, and its growing volume will make possible many useful and interesting investigations which have not yet been taken up. The actuary to an assurance company, moreover, has unfailing opportunities of usefulness in the practical study of that vitally important subject, the investment of its funds. This, in the days that lie ahead of us, will demand more

than ever watchful and well-trained handling. Of all such considerations one is not forgetful, when calling attention to the developing possibilities for actuarial science outside their borders.

But those possibilities are insistent, and important. Here again they are the result of surrounding conditions; but conditions in this case greater in their appeal than those of any particular business, conditions of the national life as a whole. Movements of thrift throughout the mass of our population have been growing for more than a century, and have been gradually coming more and more within the shelter of actuarial guidance, but the intervention of the State itself as an active participator has within the memory of us all transformed the entire situation; and the convulsion through which we are now passing as a people will have the result of intensifying the sense of collective responsibility and effort for the common good. Stronger than ever in such circumstances is the need for the help of the actuary.

In this connection we have to take account of the modern scientific study of statistics, associated primarily with such names as those of Sir Francis Galton and Professor Karl Pearson. Its rationale, and general principles, may be grasped by anyone possessing the usual mathematical training of an actuary, without exceptional difficulty; but its more elaborate developments call for some rather severe and special study. It is more or less, for the present, a science in the making; and presents an ample field to the trained investigator. Abstruse as it may seem, and in some respect is, its object, and the results it seeks, are eminently practical. The enormous importance of statistical inquiry is being more fully realized every day. Statistics lie at the base of every material aspect of our national existence. The crude methods by which they were often treated in the past, the pitfalls of which their study is full and into which so many have fallen, have made them the subject of popular suspicion, and of some very uncomplimentary proverbs. A moment's reflection, however, will show to common sense that from their accurate compilation and correct interpretation, if these can be accomplished, must follow results of the utmost human value. Thus much should certainly be clear to the actuary. The new methods for their study are on the path of true progress, and there is nothing

fuller of future potency; for as questions relevant to the condition of the nation grow in importance, so will statistical investigations increase and multiply. Here is surely a supreme opportunity for actuarial science to place itself in the main current of public usefulness. Contemplating that field of great possibility, one feels instinctively that the actuary should be there. The work that some of our number have so done and are doing we gladly acknowledge; and those of us to whom advancing years deny the mental elasticity sufficient for the enterprise look across to the promised land and say to those more fortunately circumstanced, "Go up and possess it." It is curious, and an illustration of what has been said as to tendencies being latent before changing conditions make them evident and dominant, to find Dr. Farr writing, more than sixty years ago: "The whole "commerce of the country turns on contingencies which "demand the application of scientific observation and cal-"culation; and as English agriculture has its chemists, "English commerce must, to keep pace with it, ultimately "employ actuaries to calculate the risks which are now "only roughly guessed at, and thus extend the useful sphere "of an important scientific class of men." The result of subsequent experience, while it might alter in some respects the phrasing of that statement, but confirms its principle.

To come back to our starting-point: in every profession, trade, or business there is for a man no surer source of inspiration during work which must often be hard, and—what is worse—often monotonous, than a sense of the honour and dignity of his calling. That sense, for its complete attainment, depends on two things: a worthy history in retrospect, an increasing opportunity of public service in prospect. As evidence of the first I have tried to bring together a few facts familiar to us all, the excuse for gathering them into one focus being the circumstance that our Institute's existence happens just now to have completed a rounded term of years. As evidence of the second, I point to the conditions surrounding us and the future they are likely to bring.

The actuary occupies, it seems to me, a peculiarly favourable position. On the one hand he is the exponent of a science, concerned with exact knowledge. That gives a consciousness of natural law, of first principles, of boundaries which cannot

be overstepped. This consciousness has a calming and regulating influence upon the mind. It holds in check all tendencies to fantastic speculation, to unbalanced activity, to the treacherous impulse towards seeing things as one wishes them to be, instead of as they are. It is a stronghold within which to take refuge when confronted with error or fallacy. Then, on the other hand, he has to deal with men—their multitudinous affairs, their divergent personalities. There is no more wholesome mental exercise than that. It keeps the faculties alert, corrects any tendency to pedantry, applies to any fancy for theoretical extravagance the saving touch of common sense. The philosophy evolved in the study has to be brought into the market place; to be made intelligible and convincing to the plain man. In this combination of the pursuit of scientific accuracy and the sense of public utility there is surely an ideal training. We are missionaries of warfare against the hoary and deep-rooted delusion that theory and practice are enemies; and our weapon is the translation of scientific truth into the dialect of daily intercourse. It is one thing to prepare some scheme of communal benefit which is the best that the financial conditions will allow, and another to vindicate its merits to the satisfaction of a crowd of eager participants who are profoundly convinced that it should have been much better. A differential equation and a discontented policyholder are extremely diverse difficulties. The man who can successfully negotiate both has attained something which neither alone could give him, and which is of no small value. This is what our education and experience taken together can do for us, and it is in the just balance of the academic and the popular elements that the problem of successful achievement lies. More, I venture to say, than successful achievement in any limited sense; for the man who can be equally at home in the cloistered mental seclusion of an exact thinker, and the companionship and understanding of his fellow men, has not only made a professional success: he has widened and enriched his entire life.

I should like to say just one other thing before I have done. It is, I suppose, something depending largely on individual temperament, and therefore entitled to a limited value only, as such utterances, like rivers, can rise no higher than their source; and on such terms alone it is offered.

Expressions of that kind on such occasions as these, whether they be right or wrong, may perhaps equally serve some useful end, as indications of personality. I have a very strong conviction of the value to us, as to all professional men, of sufficient touch with the general sum of what is said and thought in the world as will give us a chance of viewing our profession not only as a thing in itself, but as something which is a part of an infinitely greater whole. This is not the mere familiar theory that every man should have a hobby, though that in itself is good; and when we hear that the first actuary of one of the oldest British life offices was a brilliantly famous chess player, or that Joshua Milne had an unusually minute knowledge of natural history, and one of the best botanical libraries in London, I think we should reckon it to them for righteousness. What I mean will be better conveyed by another illustration. One of the greatest pulpit orators of the nineteenth century, when speaking on one occasion to students preparing for his profession, advised them throughout their lives assiduously to read, adding with emphasis that he did not mean exclusively theological reading. Reading, he told them, was essential for keeping the mind fresh; and although what they read might seem entirely remote from their subject, it would in unexpected ways be of constant help to them. I do not think anything much wiser has ever been said. However well a man knows his own business, I venture to say that to have the power of viewing it from without as well as from within, of regarding it as part only of the inexhaustible and unfathomable variety of the world in which we live, and considering it in its relations to that whole, is an invaluable help to his knowledge of the thing itself. It would be foolish to say that this can or should come from reading only. Men and minds differ. What some get in that way others may get from art, from travel, from the discourse of friendship. But in some form to have it, in some degree to enter into the spirit of the saying, "Homo sum; humani nihil a me alienum puto," will help us as nothing else can to that largeness of conception and breadth of view in dealing with affairs, which mere technical absorption, however excellent, tends if uncorrected to cramp or even to paralyze. For which reason I prefer to close with words not my own, the simple directness of which cannot be improved upon; the words of one long preserved to us through a benign old age, whose

presence now removed we have not yet ceased to miss from our gatherings: the words of Arthur Hutcheson Bailey, when, speaking from this chair thirty-six years ago he said,

"An actuary should be a man of general culture, with a "knowledge both of books and men, and the more he has of "both the better."

Yields on Redeemable Securities.

Although the investment yield with allowance for income-tax has long been familiar to students of the theory of compound interest, it is practically unknown to the ordinary investor, and has not, probably, been generally adopted even by actuaries.

Several causes have contributed to the yield of theory not having become the standard yield of practice. One is that many investors—a majority, perhaps, in number, although not in amount—are more interested in the immediate return on their invested capital than in the true yield, and are, consequently not much concerned as to a factor which affects only the latter. This would be a good reason for quoting yields without allowance for either profit on redemption or income-tax, but not for quoting with allowance for the one and not for the other. The fact that all investors are not liable for tax at the full rate would introduce a little difficulty, but the difficulty could be met by a simple approximate adjustment, and even without such adjustment the yield with allowance for tax at the full rate would, in most cases, be nearer the true yield than that obtained by neglecting tax altogether. In any case, the point is not one that affects the more important classes of investors such as insurance companies and trust companies.

Another contributing cause has, no doubt, been that until quite recently the allowance for income-tax did not, as a rule, make much difference in the yield, and that since it has become material it has been rather overshadowed by more important financial factors. Even now the average financial man, although he may go so far as to recognize that the fact of the profit on redemption not being subject to income-tax is a "good feature" of an investment, is inclined to regard its quantitative measurement as an "actuarial refinement."

A third cause may have been the influence of the American bond-value tables. The question of income-tax, as a factor in

the yield on securities, did not, of course, arise in the United States until quite recently, and it will not arise now—as affecting in any special way the yield on redeemable securities—if it should be held that profits on redemption are subject to tax under the U.S. law. It is not unlikely, therefore, that the American tables will be continued in their present form, and so far as they are used in the United Kingdom they will, in that case, tend to maintain the yield without allowance for tax. Incidentally, an interesting problem arises as to the yield on a redeemable security subject to both American and British tax, supposing the former to be charged both on income and increase of capital-value, and the latter on income only.

A more effective cause, perhaps, (at any rate so far as actuaries are concerned) than any of those mentioned, has been a doubt as to the theoretical basis of the yield with allowance for tax. As ordinarily calculated the yield with allowance for tax is the yield that would be realized (by an investor subject to tax at the full rate) in the event of the rate of tax remaining constant at the rate in force at the time of calculation, and of no change taking place in the mode of assessment. This is obviously unlikely in general to be the correct yield, for it is improbable that the rate of tax would remain constant throughout the term of any security except one maturing in a very short time. But it may be expected to be more nearly correct than the yield obtained on the basis of no tax at all. This will be apparent from a comparison of yields on any reasonable assumptions as to the future course of the rate of income-tax. Suppose, for example, that during the term of a security redeemable at par in n years, and bearing interest at rate g yearly, the rate of tax were to increase or decrease from t_1 to t_n in such a way that $\frac{1}{1-t_1}$, $\frac{1}{1-t_2}$, &c., were in arithmetic progression. The gross yield, at a price of $1-k$, would be roughly

$$\left[g+\frac{k}{n}\,\frac{1-\frac{1}{2}(t_1+t_n)}{(1-t_1)(1-t_n)}\right]\Big/\left[1-\frac{n+1}{2n}k\right]$$

Or suppose that with the same data the rate of tax were to increase or decrease in such a way that the net dividends were in geometric progression with a common ratio of $(1+\lambda)$. The gross yield in that case would be roughly

$$\left[g+\frac{k}{n}\cdot\frac{1}{1-t_1}\cdot\frac{\mathbf{a}'_{\overline{n}|}}{n}\right]\Big/\left[1-\frac{n+1}{2n}k\right],$$

where $\mathbf{a}'_{\overline{n}|}$ is calculated at rate λ. For practical values of t_n or λ either of these expressions would be much more nearly equal to

$$\left[g+\frac{k}{n}\cdot\frac{1}{1-t_1}\right]\Big/\left[1-\frac{n+1}{2n}k\right]$$

the approximate gross yield on the basis of a constant rate of tax t_1 than to

$$(g+k/n)\Big/\left(1-\frac{n+1}{2n}k\right)$$

the corresponding gross yield on the basis of no tax.

The possibility of a change in the mode of assessment is more important and cannot be entirely excluded. It is possible, for example, that the basis on which life assurance companies are assessed might be fundamentally altered so that calculations made on the assumed continuance of the present practice of taxing interest-earnings would no longer hold good. But any such alteration would, probably, bring compensating advantages, and the possibility of taxation being extended to profits on redemption, while interest-earnings continue to be taxed, may perhaps be disregarded. Apart from the practical difficulties which such a change would entail, it would clearly be inequitable in view of the large number of investments that have been made on the basis of the present law. Moreover, it is a reasonable assumption that if at any time profits on redemption were made liable to income-tax existing investments would be exempt—on the analogy of purchases of reversions subject to old rates of duty. This might not be entirely satisfactory—nor equitable as regards trustees and others who may be compelled to realize investments—because market-values would still be affected adversely, but it would, at any rate, enable an investor who had purchased on the basis of the existing income-tax law to realize his anticipated yield by holding his investment until maturity.

While each of the foregoing causes may have accounted, in some degree, for the yield with allowance for tax having remained more or less an academic yield, a determining cause has almost certainly been the fact that no table of yields on this basis has been published. Sufficiently simple formulas for calculating the yield with allowance for tax in individual cases have been given (*see* Text-Book, Part I, Revised Edition, pp. 118, 121, and *J.I.A.*, vol. xlix, pp. 366-369), but a formula, however simple, does not, in practice, answer quite the same purpose as a table.

It seems worth while, therefore, to consider the practicability of constructing a convenient table.

Since (with the usual notation) the net yield with allowance for tax is approximately

$$[g(1-t)+k/n] \Big/ \left[1-\frac{n+1}{2n}k\right]$$

it follows that the difference between the net yields, i_1 and i_2, corresponding to two rates of tax t_1 and t_2 would be very nearly

$$i_1 \times g(t_2-t_1)/[g(1-t_1)+k/n]$$

which—so far as k and n are concerned—depends only on the ratio k/n. It would appear therefore that a table of net yields for a standard rate of tax—say at the present time 5*s*. in the £—with a supplementary column of the values of $g(t_2-t_1)/[g(1-t_1)+k/n]$ for $t_2-t_1=$ say $\frac{1}{20}$ (corresponding to a difference of 1*s*. in the £ in the rate of tax) would give the net yield with approximate accuracy on the basis of any rate of tax that may be anticipated within the next few years. A simpler, but somewhat rougher, adjustment for a difference in the rate of tax would be

$$g(t_2-t_1) \Big/ \left(1-\frac{n+1}{2n}k\right)$$

which is practically independent of n except when n is very small and may be taken for practical purposes as equivalent to

$$g(t_2-t_1) \Big/ \left(1-\frac{1}{2}k\right).$$

As regards the table itself, the most convenient arguments in practical use would obviously be the *price* and the *unexpired term*. A table for these arguments would, however, be open to the objections that the useful values would run across the page in a comparatively narrow diagonal band, and that it would not lend itself well to interpolation. If the table were sufficiently extensive these objections might be outweighed by the great practical advantage of entering the table directly with the data. But for a table of limited extent a different arrangement would be more suitable. The approximate formula

$$[g(1-t)+k/n] \Big/ \left[1-\frac{n+1}{2n}k\right]$$

again suggests a solution. For since the yield corresponding to

a given value of g depends mainly on the value of k/n, it would appear that the arguments k/n and n would give a compact table and one in which first difference interpolation should be nearly accurate. Tables are, accordingly appended on this basis. It will be seen that the differences in both directions vary slowly, and that an interpolated value for $n+m$, $k/n+\kappa$, may be taken as approximately $i_{n,\,k/n}+\frac{m}{5}\Delta_n+10\kappa\Delta_{k/n}$. The interpolation, being of a simple nature, may be performed by inspection.

For example, the net yield allowing for tax at 5*s.* on a 3 per-cent security redeemable at par in 33 half-years, and bought at 75 (so that $k/n = {\cdot}7575$), will be approximately $4{\cdot}14+{\cdot}6\times{\cdot}11+{\cdot}575\times{\cdot}32=4{\cdot}39$; and the yield allowing for tax at 6*s.* would be approximately $4{\cdot}39-4{\cdot}0\times{\cdot}0439=4{\cdot}21$. The correct results to two places would be 4·40 and 4·22.

A table is added showing the adjustments for a difference in tax by the alternative method mentioned above. In the foregoing example the adjustment by this method for an extra 1*s.* tax would be $-{\cdot}17$, making the net yield 4·22.

Approximate net yield (convertible Half-yearly) with Allowance for Income-tax on a security redeemable at par in n half-years, bearing interest at rate g, payable half-yearly, and bought at a discount of k per-cent.

$g = {\cdot}025$.

n	NET YIELD WITH ALLOWANCE FOR TAX AT 5s.								
	k/n								
	·6	·7	·8	·9	1·0	1·1	1·2	1·3	1·4
5	3·13	3·35	3·56	3·78	4·00	4·22	4·44	4·66	4·88
10	3·18	3·41	3·64	3·87	4·11	4·35	4·59	4·83	...
15	3·24	3·48	3·72	3·97	4·23	4·49	4·76	...	...
20	3·29	3·55	3·81	4·09	4·37	4·65	4·95	...	...
25	3·35	3·63	3·91	4·21	4·51	4·83	...	...	...
30	3·42	3·71	4·02	4·34	4·68	...	...	...	...
35	3·49	3·81	4·14	4·49	4·87	...	...	...	...
40	3·56	3·91	4·27	4·66	...	...	...	...	...
45	3·64	4·01	4·42	4·85	...	...	...	...	...
50	3·73	4·14	4·58	...	...	...	...	...	...
55	3·83	4·27	4·76	...	...	...	...	...	...
60	3·93	4·42	4·97	...	...	...	...	...	...
Add or deduct % for difference of 1*s.* in tax	4·1	3·8	3·6	3·4	3·2	3·1	2·9	2·8	2·7

$g = \cdot 03.$

n	Net Yield with Allowance for Tax at 5*s*.							
	k/n							
	·5	·6	·7	·8	·9	1·0	1·1	1·2
5	3·30	3·51	3·73	3·95	4·16	4·38	4·60	4·83
10	3·34	3·57	3·80	4·03	4·27	4·51	4·75	4·99
15	3·39	3·63	3·88	4·13	4·38	4·64	4·91	...
20	3·44	3·70	3·96	4·23	4·51	4·79	...	...
25	3·49	3·77	4·05	4·34	4·64	4·96	...	...
30	3·55	3·84	4·14	4·46	4·80	5·15	...	...
35	3·61	3·92	4·25	4·60	4 97	...	...	...
40	3·67	4·01	4·36	4·74	...	...	...	...
45	3·74	4·10	4·49	4·91	...	...	...	...
50	3·81	4·20	4·63	...	...	...	...	...
55	3·89	4·31	4·78	...	...	...	...	...
60	3·97	4·43	4·95	...	...	...	...	...
Add or deduct % for difference of 1*s*. in tax	4·6	4·4	4·1	3·9	3·7	3·5	3·4	3·2

$g = \cdot 035.$

n	Net Yield with Allowance for Tax at 5*s*.								
	k/n								
	·3	·4	·5	·6	·7	·8	·9	1·0	1·1
5	3·25	3·47	3·68	3·90	4·11	4·33	4·55	4·77	4·99
10	3·28	3·50	3·73	3·96	4·19	4·43	4·66	4·90	...
15	3·31	3·54	3·78	4·03	4·28	4·53	4·79	...	...
20	3·34	3·58	3·84	4·11	4·37	4·64	4·93	...	...
25	3·37	3·63	3·90	4·18	4·47	4·77	...	...	...
30	3·40	3·67	3·96	4·26	4·58	4·90	...	...	...
35	3·43	3·72	4·03	4·35	4·69	...	...	...	...
40	3·46	3·77	4·10	4·45	4·82	...	...	...	...
45	3·50	3·83	4·18	4·56	4·97	...	...	...	...
50	3·53	3·88	4·26	4·67	...	...	...	...	...
55	3·57	3·94	4·35	4·80	...	...	...	...	...
60	3·61	4·01	4·45	4·94	...	...	...	...	...
Add or deduct % for difference of 1*s*. in tax	5·4	5·1	4·8	4·6	4·3	4·1	4·0	3·8	3·6

$g = \cdot 04.$

n	Net Yield with Allowance for Tax at 5s.							
	k/n							
	·2	·3	·4	·5	·6	·7	·8	·9
5	3·42	3·63	3·85	4·06	4·28	4·50	4·72	4·94
10	3·44	3·66	3·89	4·12	4·35	4·59	4·82	...
15	3·46	3·69	3·93	4·18	4·42	4·68	4·93	...
20	3·48	3·72	3·98	4·24	4·50	4·78	...	...
25	3·50	3·76	4·03	4·30	4·59	4·89	...	...
30	3·52	3·79	4·08	4·37	4·68	...	...	...
35	3·54	3·83	4·13	4·45	4·79	...	...	...
40	3·56	3·87	4·19	4·53	4·90	...	...	...
45	3·59	3·91	4·25	4·62	...	...	...	...
50	3·61	3·95	4·32	4·71	...	...	...	...
55	3·64	3·99	4·39	4·81	...	...	...	...
60	3·66	4·04	4·46	4·93	...	...	...	...
Add or deduct % for difference of 1*s*. in tax	5·9	5·6	5·3	5·0	4·8	4·6	4·4	4·2

$g = \cdot 045.$

n	Net Yield with Allowance for Tax at 5s.						
	k/n						
	·1	·2	·3	·4	·5	·6	·7
5	3·59	3·80	4·01	4·23	4·44	4·66	4·88
10	3·60	3·82	4·04	4·27	4·50	4·74	4·97
15	3·61	3·84	4·08	4·32	4·57	4·82	...
20	3·62	3·86	4·11	4·37	4·64	4·91	...
25	3·63	3·88	4·15	4·43	4·71	...	...
30	3·64	3·91	4·19	4·48	4·79	...	...
35	3·65	3·93	4·23	4·54	4·87	...	...
40	3·66	3·96	4·27	4·61	4·96	...	...
45	3·67	3·98	4·32	4·68	...	...	...
50	3·68	4·01	4·37	4·75	...	...	...
55	3·70	4·04	4·42	4·83	...	...	...
60	3·71	4·07	4·47	4·91	...	...	...
Add or deduct % for difference of 1*s*. in tax	6·3	6·0	5·7	5·4	5·1	4·9	4·7

$g = \cdot 05$.

n	Net Yield with Allowance for Tax at 5s.				
	k/n				
	·1	·2	·3	·4	·5
5	3·96	4·18	4·39	4·61	4·82
10	3·97	4·20	4·43	4·66	4·89
15	3·98	4·22	4·46	4·71	4·96
20	3·99	4·25	4·50	4·77	...
25	4·01	4·27	4·54	4·83	...
30	4·02	4·30	4·59	4·89	...
35	4·03	4·33	4·63	4·96	...
40	4·04	4·35	4·68	...	...
45	4·06	4·38	4·73	...	...
50	4·07	4·42	4·79	...	...
55	4·09	4·45	4·84	...	...
60	4·10	4·48	4·90	...	...
Add or deduct % for difference of 1*s.* in tax	6·3	6·0	5·8	5·5	5·3

Approximate addition to, or deduction from, net yield (with Allowance for Tax at 5*s.) corresponding to a decrease or increase of* 1*s. in the rate of tax (alternative method).*

$g = \cdot 025$		$g = \cdot 03$		$g = \cdot 035$		$g = \cdot 04$		$g = \cdot 045$		$g = \cdot 05$	
k	Addition or deduction	k	Addition or deduction	k	Addition or deduction	k	Addition or deduction	k	Addition or deduction	k	Addition or deduction
0–15	·13	0–6½	·15	0–11	·18	0–5	·20	0–8½	·23	0–4	·25
15–27½	·14	6½–18	·16	11–20½	·19	5–14	·21	8½–16½	·24	4–11	·26
27½–39	·15	18–29	·17	20½–29	·20	14–22	·22	16½–23½	·25	11–18	·27
39–48½	·16	29–38	·18	29–37	·21	22–30	·23	23½–30	·26	18–24½	·28
...	...	38–46	·19	...	...	...	...	...	...	...	...

Investments a Hundred Years Ago. *By* W. Palin Elderton, F.I.A.

"It is a favourite maxim of mine that history, while it should be scientific "in its method, should pursue a practical object. That is, it should not merely "gratify the reader's curiosity about the past, but modify his view of the present "and his forecast of the future."—J. R. Seeley, "Expansion of England."

It is, perhaps, worth while at the present time to go back to the earlier records of insurance with the hope of finding out how the old masters of insurance management faced the problems which arose out of fluctuations in the price of securities due to international political events. Every insurance generation has its fashion in investments, but the main principles will always be much the same, so that even at the present time it may be possible to get some help, or at any rate some interest, from the study of the investments of life assurance offices made many years ago. It has become the fashion to invest in various classes of securities instead of to restrict investments to British Government funds and mortgages: but owing to the wish to help to finance the country in the present difficulties, and partly owing to the anxiety of the Government itself in this connection which caused it to ask insurance companies to make somewhat substantial investments in War Loan, insurance companies at the present time are far more nearly in the position of having their funds mainly in Government securities than they have been for more than fifty years. In looking back into past records it is difficult to get facts which are suitable for investigation. Few life insurance companies had funds sufficiently large a hundred years ago to make their investments the matter of extreme importance which they now appear to us to be. There was, however, one exception to this general rule, namely the "Society for the Equitable Assurances on Lives and Survivorships," and the present notes relate merely to an examination of the investment policy of that Society: this is, perhaps, especially interesting because its affairs were controlled during the difficult times at the end of the eighteenth and beginning of the nineteenth centuries by a man who might be described as the father of practical actuarial work from the standpoint of life assurance.

As the funds of the Old Equitable Society in 1810 were over three millions and were increasing by £300,000 a year, the problem of investment had assumed considerable proportions. Under its original deed of settlement the Society had taken what seem

to be fairly wide powers of investment, for the clause governing its investments reads as follows:

> " . . . . . the premiums of assurance paid by the
> " Members of the said Society (except so much of the said
> " premiums as shall, by a General Court of the said Society,
> " be judged necessary for the current expenses of the said
> " Society) shall be laid out in Government or other good and
> " sufficient securities."

Although one would have expected from the wording of this clause that the Society could have invested in various classes of securities, it confined its investments during the early years entirely to Government securities, and as it published this fact in the short account of the Society which served as a prospectus, it seems to have been felt necessary in 1778 to put the matter before the members, and they passed a bye-law which begins by reciting that it has been "the usage of the Directors to invest "the monies of the Society altogether in the Public Funds, and "this is accordingly published in the succinct account of the "Plan and Basis of the Society as the usual practice", and it goes on to say that nevertheless the Directors have full and free liberty to invest in accordance with the clause in the original Deed, and adds that "it may be desirable that some of the "monies should be laid out upon different securities, presumably "of equal validity, rather than that the whole should rest upon "one and the same security." Here the matter seems to have rested until 1858, when a bye-law was passed giving the Directors power to lend on policies (in this connection it may be mentioned that they had received legal opinion from Edward B. Sugden, J. Campbell, and A. R. Sidebottom in 1831 that it was impossible for them to do this). In 1865, in 1876, and on subsequent occasions more definite powers were given, but we need not dwell upon these in the present connection. It is probable that the Society limited its investments to Government Annuities in the first instance because the Directors, as trustees, did not feel that they had power to invest in anything else, and even after the somewhat indefinite bye-law of 1778 they may have been influenced by similar considerations.

We may now turn to what happened in the actual practice of investment, and from the earliest accounts it seems that the Society had, in 1767, only £6,000 in the 3 per-cent Consolidated Annuities, and £4,000 in 3 per-cent Anno 1726: £2,000 of the former having been purchased in that year at $86\frac{3}{4}$ and £1,000 of

the latter at 88½. At these times the funds were growing very slowly because more than half the business was for a term of one year or for a short term. It may be of interest in passing to mention that the Society borrowed £650 at 5 per-cent from Charles Gould (afterwards Sir Charles Morgan), who was then Vice-President of the Society, to enable it to pay a claim for that amount "without selling out some of the stocks still remaining "in the Old Trustees' Names."

From 1770 onwards the funds of the Society seem to have grown steadily, and in 1772 the various Government securities amounted to just under £30,000, and were made up of old South Sea Annuities, Consolidated Annuities, Anno 1726 3 per-cent and a small amount of new South Sea holdings at 3 per-cent. All the holdings were valued at 86½, which seems to be about one point lower than the amount paid for stock purchased during the year in question: but it is impossible at this distance of time to estimate the extent to which accrued interest may have appeared in the prices paid for securities as stated in the accounts. In 1778 the Society sold £20,000 of stock out of its holding of 3 per-cent Consolidated Bank Annuity "in order to purchase other stock": a point worth mentioning if only because it shows, as, in fact, do many subsequent dealings, that the idea that an insurance company may refrain from looking for opportunities of making profitable sales, was not the view of these pioneers of life insurance.

In 1779 mortgages appear in the accounts for the first time, and it may be of interest to mention that one of the mortgages now on the books of the Society was granted originally in June, 1798, although it has, of course, in the meantime had to pass through the vicissitudes of many re-arrangements. From 1779 onwards the new investments were divided between mortgages and investments in the Public Funds, and at the beginning of the nineteenth century about £400,000 was invested in mortgages and the holdings in the 3 per-cent and 4 per-cent Annuities amounted to £1,290,000 and £400,000 respectively.

It was during the closing years of the eighteenth century and the early part of the nineteenth century that the funds of the Society were growing most rapidly, and evidence of this is found in the large new investments which were made by the office during these years. It frequently appears that several hundred thousand pounds of new investments were made, and as the Society was granting new assurances to the extent of a million of

sum assured, which was far more than had been obtained in previous years, it was clear that the funds of the Society would be expected to increase for a considerable period. At the end of 1816 however, the Society adopted a bye-law which, by limiting to 5,000 the number of policies participating in profits, had the effect of discouraging new insurances, and although the funds continued to increase for several years, the position gradually underwent a change, and after the funds had been over nine millions in 1840, they gradually dropped until they were about four millions in 1870.

The following table shows the investments of the Society on various occasions, and may give some idea of the changes that took place in the investments that had been made.

Table showing Investments (in £1,000's) of Old Equitable.

Year	Government Stocks			Exchequer Bills	Mortgages
	3 %	4 %	3½ %		
1773	34	...	...	...	...
1793	357	120	...	...	197
1803	1,290	400	...	...	418
1805	1,905	400	...	...	412
1810	3,540	400	...	19	402
1815	5,520	400	...	...	562
1820	7,800	400	...	...	729
1825	9,005	...	380	...	1,157
1830	8,130	...	400	...	2,181
1835	7,750	...	400	60	2,812

In the fifteen years from 1779, when mortgages were first granted, until 1793, the sum laid out on mortgage reached £197,000, and by 1799 it reached £400,000, at which figure it remained for many years. Stocks were high during the early period covered by our table, but from 1793 onwards prices fell and between that date and 1805 the prices of 3 per-cent Government stocks were sometimes below 50, nearly always below 60, and rarely above 70. During the following ten years prices were a little higher, but still low, and while they were never below 55 they were never above 73. After 1815 prices improved, and from 1828 onwards 3 per-cent stocks were almost always above 80 and frequently above 90.

If these variations in the value of the Government stocks are

compared with the changes in the investments indicated by our table, it will be seen that the amount invested in mortgages was not increased during the time when Government stocks were so low that profitable investments could be made, but in the early period before the low prices became established, and in the later period, substantial investments were made in mortgages. Large amounts of securities were sold between 1825 and 1840 in order to increase the amount advanced on mortgage. It will also be observed that investments were made in Exchequer Bills as well as in the more permanent type of security, and these seem to have been bought and sold from time to time so that they were apparently treated as a convenient temporary investment.

The point of view of the men responsible for the management in these early days was that when the price of securities was high it was wise to put a substantial amount on mortgage and some investments in short-dated securities: but when a good return could be obtained from permanent investments in the Public Funds mortgages were neglected. In making this remark I may, perhaps, be accused of reading into past actions thoughts which were not in the minds of those of whom I am writing. But there is evidence in some of the addresses of William Morgan, and in his little book on the early history of the Old Equitable, that he at any rate appreciated these points and it was partly his appreciation of them that made him fearful lest the Society to which he had devoted so much energy should suffer from having to make investments less profitably, and it is, I think, clear that this was one of the reasons which led him to advise the Directors and members to restrict the new business so that such profits as had been due to temporary advantages, which might not recur, would not have to be spread over so large a number of cases as would have necessarily followed from the continuance of the large amount of new business which the Society had been obtaining from 1800.

If Seeley's maxim that history should pursue a practical object is one that it is wise for us to follow, we may endeavour to see what can be learnt from the information we have obtained.

The most obvious conclusion to be drawn is that it is possible to frame and follow a consistent investment policy which, over a period of years will avoid the dangers of depreciation and take advantage of appreciation. Possibly another conclusion which

ought to be drawn is that an insurance company can make satisfactory financial arrangements even if it has a very restricted field of investment. When set out in this way these conclusions seem obvious : but in any attempt to apply them in practice it is necessary to face the difficulty involved in deciding whether we are passing through a period of low prices or are merely at the beginning of a long fall in prices. If at the present time we knew that prices of securities would, on the average, fall 15 or 20 per-cent in the next five years and then rise more or less steadily there would be little difficulty in applying general principles. But the problem in practice involves an estimate of the course of future prices and, while we might be prepared to commit ourselves to a broad statement that no permanent rise can take place till some little time after peace and therefore to justify the application of some settled policy over a long period, we are haunted by the fear that a small misjudgment of the course of prices, while unimportant over a long term of years, may place us in a difficult position as regards a few balance sheets in the meantime. In this connection it may be of interest to record that the practice of the Old Equitable was to take market prices when valuing securities—I think prices must have been taken at 1 November instead of 31 December, when the valuation was nominally made, as all valuations were based on assurances existing at 1 November on the assumption that they would remain in force till the end of the year. Morgan referred to this practice in his addresses of 1825 and 1829 when the Society was investing in mortgage whenever possible, and warned members not to look on the existing prices as permanent and not to think of surplus in terms of inflated values. However this may be, it is clear that the method of stating values of securities a hundred years ago was optional, while at the present time we are restricted in a way that may operate severely when funds depreciate even from a temporary cause. Such temporary effects cannot be foretold in advance, but if we can follow, as it seems to me Morgan and his Directors followed, an investment policy based on a broad outlook over a number of years, we might well run the risk of temporary disappointments.

At the present time, no doubt, everyone responsible for the investment of life assurance funds is trying to read rightly the trend of prices and to base his action on his opinion. I am doubtful if modern practice will yield many examples of greater foresight than that of the men responsible for investing life

insurance funds about a hundred years ago. It seems to me that Morgan and his Directors made and followed a consistent policy in times which bear some likeness to our own, and the story of their efforts may be useful to us in deciding on our present action and in making our forecast of the future.

A General Expansion Formula.

De Morgan wrote at considerable length, and almost with enthusiasm, on Arbogast's Method of Derivations, which is virtually an extension of Taylor's Theorem. He says, "Few, even among mathematicians, are aware of the "power of this process .... which well deserves to be made "as common as Taylor's Theorem.... By help of the method, "expansions which analysts usually avoid as much as possible, "at almost any expense of circumoperation, are carried on "with the greatest facility even further than is necessary....

"The foundation of Arbogast's methods is a contrivance "for expediting the expansion of $\phi(a+bx+cx^2+\ldots)$ into a "series of the form $A+Bx+Cx^2+\ldots$" De Morgan gives materials for proceeding as far as the term $\ldots x^{12}$; "not that "so much will often be necessary, but because it is desirable to "show with how little trouble questions of enormous labour in "the ordinary way ... may be looked upon without dismay."

Readers who desire to study the mathematical basis of the method, or to go so far as x^{12} in their expansions, should refer to De Morgan's Differential and Integral Calculus, pp. 328–335; also to the article from his pen in the English Cyclopædia (title, "Taylor's Theorem"), and in the *Journal*, vol. xii, p. 206. He works out the matter very fully, and discusses the relation between Arbogast's process and the Multinomial Theorem when $\phi(u)$ is u^n. Reference may also be made to a brief discussion of the method in Williamson's Differential Calculus. The object of the present Note is to give the formula in its general form, and also in some specific cases, as far as the term involving x^6, which will be amply sufficient for most practical purposes.*

* Those who wish to check or extend the formula by means of De Morgan's tables should notice that he omits d; *i.e.*, our $a+bx+cx^2+dx^3+ex^4\ldots$ is his $a+bx+cx^2+ex^3+fx^4\ldots$ so that after c each of his letters must be replaced by the preceding one.

I.—*General Formula.*—If $\phi(a+bx+cx^2+\ldots\ldots)=A+Bx+Cx^2+\ldots\ldots$, then denoting the successive differential coefficients of $\phi(a)$ with respect to a by ϕ', ϕ'', ϕ''', &c.

$$A=\phi(a)=\phi$$

$$B=b\phi'$$

$$C=c\phi'+\frac{b^2}{2}\phi''$$

$$D=d\phi'+bc\phi''+\frac{b^3}{6}\phi'''$$

$$E=e\phi'+\left(bd+\frac{c^2}{2}\right)\phi''+\frac{b^2c}{2}\phi'''+\frac{b^4}{24}\phi^{\mathrm{IV}}$$

$$F=f\phi'+(be+cd)\phi''+\left(\frac{b^2d}{2}+\frac{bc^2}{2}\right)\phi'''+\frac{b^3c}{6}\phi^{\mathrm{IV}}+\frac{b^5}{120}\phi^{\mathrm{V}}$$

$$G=g\phi'+\left(bf+ce+\frac{d^2}{2}\right)\phi''+\left(\frac{b^2e}{2}+bcd+\frac{c^3}{6}\right)\phi'''$$
$$+\left(\frac{b^3d}{6}+\frac{b^2c^2}{4}\right)\phi^{\mathrm{IV}}+\frac{b^4c}{24}\phi^{\mathrm{V}}+\frac{b^6}{720}\phi^{\mathrm{VI}}$$

It will be observed that in every case the coefficient of ϕ' is of the first dimension, that of ϕ'' is of the second dimension, and so on.

The above coefficients may be written down at once from De Morgan's tables based on Arbogast's theory, and in the same way the series may be continued to any extent. But without reference to these tables, the series may be very simply derived (to the extent to which we have here used it) by the following method (*Cf.* De Morgan, *J.I.A.*, vol. xii, p. 212). Let $b+cx+dx^2+\ldots=z$. Then, by Taylor's Theorem,

$$\phi(a+bx+cx^2\ldots)=\phi(a+xz)=\phi(a)+xz\,.\,\phi'(a)+x^2z^2\,.\,\phi''(a)/2\,!+\ldots$$

and if we expand z^2 as far as x^4, z^3 as far as x^3 and so on, by ordinary multiplication, and then collect coefficients of x, x^2, $x^3\ldots x^6$, we shall arrive at the above values for A, B, C . . . G.

II.—*Exponential Function.*—When $\phi(u)=e^u$, $\phi, \phi', \phi''\ldots$ are all equal to e^a. Hence $e^{a+bx+cx^2+\ldots}$

$$=e^a\Big[1+bx+\Big(c+\frac{b^2}{2}\Big)x^2+\Big(d+bc+\frac{b^3}{6}\Big)x^3+\Big(e+bd+\frac{c^2}{2}+\frac{b^2c}{2}+\frac{b^4}{24}\Big)x^4$$

$$+\Big(f+be+cd+\frac{b^2d}{2}+\frac{bc^2}{2}+\frac{b^3c}{6}+\frac{b^5}{120}\Big)x^5$$

$$+\Big(g+bf+ce+\frac{d^2}{2}+\frac{b^2e}{2}+bcd+\frac{c^3}{6}+\frac{b^3d}{6}+\frac{b^2c^2}{4}+\frac{b^4c}{24}+\frac{b^6}{720}\Big)x^6$$

$$+\ \ldots\ldots\ldots\Big]$$

III.—*Logarithmic Function.*—When $\phi(u)=\log_e u$, $\phi(a)=\log_e a$, $\phi'=\frac{1}{a}$, $\phi''=-\frac{1}{a^2}$, $\phi'''=\frac{2}{a^3}$, $\phi^{\text{IV}}=-\frac{6}{a^4}$, $\phi^{\text{V}}=\frac{24}{a^5}$, $\phi^{\text{VI}}=-\frac{120}{a^6}$, ... Hence, $\log_e(a+bx+cx^2+\ \ldots)$

$$=\log_e a+\frac{b}{a}x+\Big(\frac{c}{a}-\frac{b^2}{2a^2}\Big)x^2+\Big(\frac{d}{a}-\frac{bc}{a^2}+\frac{b^3}{3a^3}\Big)x^3$$

$$+\Big(\frac{e}{a}-\frac{2bd+c^2}{2a^2}+\frac{b^2c}{a^3}-\frac{b^4}{4a^4}\Big)x^4$$

$$+\Big(\frac{f}{a}-\frac{be+cd}{a^2}+\frac{b^2d+bc^2}{a^3}-\frac{b^3c}{a^4}+\frac{b^5}{5a^5}\Big)x^5$$

$$+\Big(\frac{g}{a}-\frac{2bf+2ce+d^2}{2a^2}+\frac{3b^2e+6bcd+c^3}{3a^3}-\frac{2b^3d+3b^2c^2}{2a^4}+\frac{b^4c}{a^5}-\frac{b^6}{6a^6}\Big)x^6$$

$$+\ \ldots\ldots\ldots$$

IV.—*Power Function.* (*Multinomial Theorem*).—When $\phi(u)=u^n$, $\phi(a)=a^n$, $\phi'=na^{n-1}$, $\phi''=n(n-1)a^{n-2}$, $\phi'''=n(n-1)(n-2)a^{n-3}$, and so on. Hence, if we write C_r for $\frac{n!}{r!\,(n-r)!}$, the ordinary binomial coefficient of x^r, we have $(a+bx+cx^2+\ \ldots)^n$

$$=a^n+nba^{n-1}x+(nca^{n-1}+C_2b^2a^{n-2})x^2$$
$$+(nda^{n-1}+C_2\,.\,2bca^{n-2}+C_3b^3a^{n-3})x^3$$

$$+[nea^{n-1}+C_2(2bd+c^2)a^{n-2}+C_3\,.\,3b^2ca^{n-3}+C_4b^4a^{n-4}]x^4$$

$$+[nfa^{n-1}+C_2\,.\,2(be+cd)a^{n-2}+C_3\,.\,3(b^2d+bc^2)a^{n-3}$$
$$+C_4\,.\,4b^3ca^{n-4}+C_5b^5a^{n-5}]x^5$$

$$+[nga^{n-1}+C_2(2bf+2ce+d^2)a^{n-2}+C_3(3b^2e+6bcd+c^3)a^{n-3}$$
$$+C_4(4b^3d+6b^2c^2)a^{n-4}+C_5\,.\,5b^4ca^{n-5}+C_6b^6a^{n-6}]x^6$$

$$+\ \ldots\ldots\ldots$$

V.—*Reciprocal of Function.*—Putting $n=-1$ in the preceding formula, we have $(a+bx+cx^2+\ldots)^{-1}$

$$=\frac{1}{a}-\frac{b}{a^2}x-\left(\frac{c}{a^2}-\frac{b^2}{a^3}\right)x^2-\left(\frac{d}{a^2}-\frac{2bc}{a^3}+\frac{b^3}{a^4}\right)x^3$$

$$-\left(\frac{e}{a^2}-\frac{2bd+c^2}{a^3}+\frac{3b^2c}{a^4}-\frac{b^4}{a^5}\right)x^4$$

$$-\left(\frac{f}{a^2}-\frac{2be+2cd}{a^3}+\frac{3b^2d+3bc^2}{a^4}-\frac{4b^3c}{a^5}+\frac{b^5}{a^6}\right)x^5$$

$$-\left(\frac{g}{a^2}-\frac{2bf+2ce+d^2}{a^3}+\frac{3b^2e+6bcd+c^3}{a^4}-\frac{4b^3d+6b^2c^2}{a^5}+\frac{5b^4c}{a^6}-\frac{b^6}{a^7}\right)x^6$$

$$-\ \ldots\ldots\ldots$$

VI.—*Reversion of Series.*—If $y=ax+bx^2+cx^3+\ldots$, then

$$x=\frac{y}{a}-b\,.\frac{y^2}{a^3}+(2b^2-ac)\frac{y^3}{a^5}-(5b^3-5abc+a^2d)\frac{y^4}{a^7}$$

$$+(7b^2.\overline{2b^2-3ac}+3a^2.\overline{2bd+c^2}-a^3e)\frac{y^5}{a^9}$$

$$-(42b^3.\overline{b^2-2ac}+28a^2.\overline{b^2d+bc^2}-7a^3.\overline{be+cd}+a^4f)\frac{y^6}{a^{11}}$$

$$+\ \ldots\ldots\ldots$$

This series is not directly derivable from the general formula in Section I, but it is known that if $y=xf(x)$,

$$x=\frac{y}{f(0)}+\sum_{n=2}^{n=\infty}\left[\frac{y^n}{n\,!}\frac{d^{n-1}}{dx^{n-1}}\frac{1}{[f(x)]^n}\right]_{x=0}$$

which, in this case becomes (since $f(x)$ here is $a+bx+cx^2\ldots$)

$$y/a+\sum_{n=2}^{n=\infty}\left[\frac{y^n}{n\,!}\cdot\frac{d^{n-1}}{dx^{n-1}}(a+bx+cx^2+\ldots)^{-n}\right]_{x=0}$$

and it may easily be seen that in this expansion the coefficient of y^n is $1/n$ of the coefficient of x^{n-1} in the expansion of $(a+bx+cx^2+\ldots)^{-n}$. These latter coefficients may readily be written down by the aid of the formula in Section IV.

If the original series for y in terms of x be convergent within any range of values of x, the reverted series for x in terms of y will be convergent within *some* range of values of y; though it may be difficult or impossible to say what that

range is if the general law of the coefficients in the reverted series is not ascertainable. It must not be assumed that the series gives a unique value of x corresponding to any particular value of y: there may be more than one such value of x. For example, if y is $\sin x$ or $\cos x$ there will be an infinite number of real values of x for any given value of y: and in the simple case where y is a quadratic expression in x, say $y=ax+bx^2$, there will be two real values of x for any value of y if $\frac{y}{b}>-\frac{a^2}{4b^2}$ and no real value if $\frac{y}{b}<-\frac{a^2}{4b^2}$. In the latter case the series will evidently be divergent. In general, the reverted series gives the value of x which tends to zero with y.

VII.—*General Formula (alternative form).*—The following form may sometimes be more convenient than that given in Section I. If

$$\phi(a+bx+c\,.\,x^2/2\,!+dx^3/3\,!+\ldots)=\mathrm{A}+\mathrm{B}x+\mathrm{C}\,.\,x^2/2\,!+\mathrm{D}\,.\,x^3/3\,!+\ldots$$

Then

$$\begin{aligned}
\mathrm{A}&=\phi(a)=\phi\\
\mathrm{B}&=b\phi'\\
\mathrm{C}&=c\phi'+b^2\phi''\\
\mathrm{D}&=d\phi'+3bc\phi''+b^3\phi'''\\
\mathrm{E}&=e\phi'+(4bd+3c^2)\phi''+6b^2c\phi'''+b^4\phi^{\mathrm{IV}}\\
\mathrm{F}&=f\phi'+5(be+2cd)\phi''+5(2b^2d+3bc^2)\phi'''+10b^3c\phi^{\mathrm{IV}}+b^5\phi^{\mathrm{V}}\\
\mathrm{G}&=g\phi'+(6bf+15ce+10d^2)\phi''+15(b^2e+4bcd+c^3)\phi'''\\
&\qquad+5(4b^3d+9b^2c^2)\phi^{\mathrm{IV}}+15b^4c\phi^{\mathrm{V}}+b^6\phi^{\mathrm{VI}}
\end{aligned}$$

See De Morgan, Differential and Integral Calculus, pp. 774–6.

VIII.—*Expression of Coefficients in Determinant form.*—The writer is indebted to Professor E. T. Whittaker, F.R.S., for the remark that in many important cases the coefficients can be expressed as determinants, which show very clearly the law of the series: and (to those who are familiar with the practical rules for the evaluation of determinants) it is much easier to compute the determinants numerically than to compute the coefficients from the expanded algebraical expressions. A few examples are given below.

Professor Whittaker informs the writer that formulæ (iii) and (iv), which were the first of the determinantal formulæ to be discovered, were given by Spottiswoode and

Faure in 1853–55. Formula (ii) is due to Faà di Bruno (1855), and the corresponding formula (v) for the logarithm to Glaisher (1879). Formula (i) belongs to the same class, but appears to have been given explicitly first by Segar (1892). Formula (vi) is a combination of (i) with Burmann's theorem, and has been given as such by Professor Whittaker in his lectures, but has not previously been published.

(i) $(a+bx+cx^2+dx^3+\ldots)^n$

$$=a^n+na^{n-1}bx+\frac{x^2}{2!\;a^{2-n}}\begin{vmatrix} nb & -a \\ 2nc & (n-1)b \end{vmatrix}$$

$$+\frac{x^3}{3!\;a^{3-n}}\begin{vmatrix} nb & -a & 0 \\ 2nc & (n-1)b & -2a \\ 3nd & (2n-1)c & (n-2)b \end{vmatrix}+\ldots$$

and the coefficient of x^r is

$$\frac{1}{r!\;a^{r-n}}\begin{vmatrix} nb & -a & 0 & \ldots \\ 2nc & (n-1)b & -2a & \ldots \\ 3nd & (2n-1)c & (n-2)b & \ldots \\ 4ne & (3n-1)d & (2n-2)c & \ldots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

where the determinant has r rows and r columns, and the law of its formation is obvious.

$$\text{(ii)}\quad e^{a+bx+cx^2+\ldots}=e^a\left[1+bx+\frac{x^2}{2!}\begin{vmatrix} b & -1 \\ 2c & b \end{vmatrix}+\frac{x^3}{3!}\begin{vmatrix} b & -1 & 0 \\ 2c & b & -2 \\ 3d & 2c & b \end{vmatrix}\right.$$

$$\left.+\frac{x^4}{4!}\begin{vmatrix} b & -1 & 0 & 0 \\ 2c & b & -2 & 0 \\ 3d & 2c & b & -3 \\ 4e & 3d & 2c & b \end{vmatrix}+\ldots\right]$$

$$\text{(iii)}\quad \frac{\alpha+\beta x+\gamma x^2+\ldots}{a+bx+cx^2+\ldots}=\frac{\alpha}{a}+\frac{x}{a^2}\begin{vmatrix} a & \alpha \\ b & \beta \end{vmatrix}$$

$$+\frac{x^2}{a^3}\begin{vmatrix} a & 0 & \alpha \\ b & a & \beta \\ c & b & \gamma \end{vmatrix}+\frac{x^3}{a^4}\begin{vmatrix} a & 0 & 0 & \alpha \\ b & a & 0 & \beta \\ c & b & a & \gamma \\ d & c & b & \delta \end{vmatrix}+\ldots$$

(iv) Putting $\alpha=1$, $\beta=\gamma=\delta\ldots\ldots=0$, we get from (iii)

$$(a+bx+cx^2+\ldots)^{-1}=\frac{1}{a}-\frac{b}{a^2}x+\frac{x^2}{a^3}\begin{vmatrix} b & a \\ c & b\end{vmatrix}-\frac{x^3}{a^4}\begin{vmatrix} b & a & 0 \\ c & b & a \\ d & c & b\end{vmatrix}+\ldots$$

(v) $\log(1+ax+bx^2+\ldots)=$

$$ax-\frac{x^2}{2}\begin{vmatrix} a & 1 \\ 2b & a\end{vmatrix}+\frac{x^3}{3}\begin{vmatrix} a & 1 & 0 \\ 2b & a & 1 \\ 3c & b & a\end{vmatrix}-\frac{x^4}{4}\begin{vmatrix} a & 1 & 0 & 0 \\ 2b & a & 1 & 0 \\ 3c & b & a & 1 \\ 4d & c & b & a\end{vmatrix}$$

$$+\ldots+\frac{(-1)^{n-1}x^n}{n}\begin{vmatrix} a & 1 & 0 & 0 & 0\ldots \\ 2b & a & 1 & 0 & 0\ldots \\ 3c & b & a & 1 & 0\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \multicolumn{5}{l}{n \text{ rows and cols.}}\end{vmatrix}\cdots$$

(vi) If $y=ax+bx^2+cx^3+\ldots$, the coefficient of y^n in the expansion of x in powers of y is (*see* Section VI) $1/n$ of the coefficient of x^{n-1} in the expansion of $(a+bx+cx^2+\ldots)^{-n}$. Hence, writing down these coefficients by means of (i), and re-arranging signs:

If $y=ax+bx^2+cx^3+\ldots$

$$x=\frac{y}{a}-b\frac{y^2}{a^3}+\frac{y^3}{3!\,a^5}\begin{vmatrix} 3b & a \\ 6c & 4b\end{vmatrix}-\frac{y^4}{4!\,a^7}\begin{vmatrix} 4b & a & 0 \\ 8c & 5b & 2a \\ 12d & 9c & 6b\end{vmatrix}+\ldots$$

$$+\frac{(-1)^{n-1}y^n}{n!\,a^{2n-1}}\begin{vmatrix} nb & a & 0 & 0 & 0\ldots \\ 2nc & (n+1)b & 2a & 0 & 0\ldots \\ 3nd & (2n+1)c & (n+2)b & 3a & 0\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots\end{vmatrix}-\ldots$$

The determinants in (vi) can be simplified before evaluation, for the first column can be divided by n, and then the bottom row by $(n-1)$.

It is hoped that the above results will be of occasional use to some who may not have access to the works cited.

G. J. L.

LEGAL NOTES.

By William Charles Sharman, F.I.A., *Barrister-at-Law.*

Agent's knowledge of material facts not given in proposal.

A FURTHER case on the vexed question as to whether the knowledge of the agent is deemed to be the knowledge of the Company was that of *Ayrey* v. *British Legal, &c., Assurance Company*, 34 T.L.R. 111. This was an appeal before A. T. Lawrence and Atkin, J.J., against a decision in the Blackpool and Fleetwood County Court.

The appellant was the executrix of one Alexander McKenzie, who had effected an insurance on his life by a policy dated 14 November 1914. She claimed to recover the amount due under the policy, but the defendants refused to pay, alleging that Alexander McKenzie had made misrepresentations on his proposal form, and had withheld material information. The misrepresentations and concealment alleged were a statement that he was a fisherman, whereas, in fact, besides being a fisherman he was a member of the Royal Naval Reserve, and at the time of making the proposal he had been called up for service.

The insured expected to go mine-sweeping, and while waiting to do so he disappeared; it was supposed that he must have fallen overboard.

At the trial evidence was given to the effect that the agent of the defendant company knew that McKenzie was going mine-sweeping, and that the agent so informed the superintendent, who was district manager of the defendants; and the superintendent said that the defendants would be willing to pay on the policy, in spite of the fact that the insured was going mine-sweeping. The plaintiff further contended that as the defendant's superintendent had accepted premiums after he had full knowledge of the facts they could not now refuse to pay.

The County Court Judge decided that there was no evidence that the agent who took the proposal ever communicated the facts to the defendants before the issue of the policy, and he therefore held that the policy was void *ab initio*, and that the receipt of premiums by the superintendent after the issue of the policy did not bind the defendants. He therefore gave judgment for the defendants.

Mr. Justice Lawrence said he did not agree with the conclusion of the County Court Judge.

The assured was a fisherman, as stated on the form, but he was also called up in the Royal Naval Reserve, a fact which was not stated on the form, and the question was whether his having been called up without the statement of that fact vitiated the policy. In his opinion it did not. True, there was a provision on the form saying that omission or concealment of material facts would render the policy void. The district manager was, however, told that McKenzie was in the Royal Naval Reserve, and it was only reasonable to suppose that telling the manager was equivalent to telling the head office. Communications made to such a person would be assumed by any reasonable assured person to be communicated to the head office. The manager had no power to make a new contract, but the old contract subsisted, and any objection that might have been taken was waived by the subsequent receipt of premiums. The appeal must therefore be allowed.

Mr. Justice Atkin delivered judgment to the same effect.

Rights of Morgagees under Courts (Emergency Powers) Act.

The case of *In re Jobson's* application and in the matter of *Chapman's* Mortgage has attracted considerable notice, owing to the rules laid down by Mr. Justice Eve in the course of his judgment regarding the rights of mortagees as modified by emergency legislation. These rules are of peculiar interest to those life assurance officials who have to deal with mortgages and are sometimes faced with the problems dealt with by the learned Judge. The case has not yet appeared in the Reports, but will be found in *The Times* of 17 January 1918.

The facts are as follows:

A mortgage was created in July 1906 on premises in Eastbourne, which were valued for the purpose at £21,800. The amount advanced was £12,500 with interest at 4 per-cent. and the money was used as capital by the mortgagors, who were traders occupying a considerable portion of the mortgaged premises. Notice to pay off the mortgage was served on the mortgagors in March 1917, and a summons for leave to realize the security under the Courts (Emergency Powers) Act was issued in July 1917. All the covenants and conditions of the mortgage deed had been duly performed and observed, and the principal

money due had been voluntarily reduced by annual payments to £8,350.

Judgment was given extending the time for the repayment of the mortgage debt, on condition that the mortgagor undertook to pay 5 per-cent interest and to repay out of principal £500 in October. In the course of his judgment Mr. Justice Eve said: "Experience has shown that considerable misconception exists as to the purpose and effect of the emergency "legislation with which this application is concerned. The Act "is not for the relief of insolvent debtors, still less for the "conscription for the benefit of debtors of the property of their "creditors. It is, as its name implies, an Act designed and "intended to meet some of the emergencies to which the "present state of war has given rise, and the particular "emergency to which the sections with which I have to deal "are directed is the inability of a debtor to discharge his "obligation on the date when it falls to be discharged. In all "cases of this class to which the Act applies it restricts the "right of the creditor to resort to his legal remedies for "enforcing payment to the extent that it forbids him the "exercise of any such remedy until after an application has "been made to the Court for leave to exercise it. Then, by "Sub-section (2), it confers on the Court an absolute discretion "in dealing with the application, subject to the limitation that "the discretion is only to be exercised in favour of a defaulting "debtor if the Court is of opinion that time should be given to "him on the ground that he is unable immediately to make the "payment by reason of circumstances attributable, directly or "indirectly, to the war. When the Court is satisfied on this "point it has power to suspend the question of any remedy for "such time, and subject to such conditions, as it thinks fit.

"The present application is one by mortgagees whose "mortgage debt became repayable in June last, and who apply "for leave to exercise any right or powers vested in them as "mortgagees to realize their security, and in what I am about "to say I am confining myself to applications of a similar "character, *i.e.*, applications by mortgagees against defaulting "mortgagors. In the ordinary way, where a mortgage debt is "called in, the mortgagor meets the demand in one of three "ways: (1) he realizes the security and pays off the mortgagee "out of the proceeds; or (2) he provides the money out of his "other resources, and takes a reconveyance; or (3) he gets the

" assistance of another lender, who pays off the mortgage and " takes a transfer as security. The last of these courses is the " one most usually adopted, and, indeed, it is almost inevitable " where the advance is of a large amount, which is treated as " capital more or less permanently borrowed for business " purposes at a fixed and moderate rate of interest.

" In considering, therefore, whether the mortgagor qualifies " himself for the exercise in his favour of the discretion conferred " by Section 2 the Court ought, in the first place, to direct its " attention to the question how, in the ordinary course, the " particular security would be dealt with. A man who had " purchased property as an investment and borrowed a portion " of the purchase-money on mortgage, might—where the " mortgage money was called in—elect either to realize that " investment and pay off the mortgage or he might prefer to " realize other investments and pay it off, or he might just as " likely determine to retain his investment and find a transferee " of the mortgage. On the other hand, a man who had borrowed " money on the security of property in his own occupation " would probably be limited to the alternatives of paying the " debt out of his other resources or of obtaining a transferee, " and a man who had raised money on the security of his " business premises and had employed the money in his business " would, almost of necessity, be compelled to meet the " mortgagee's requirements by obtaining a transferee. Nor " would these more or less personal considerations alone " determine the mortgagor's action. A more potent factor " would be the value of security in relation to the amount " of mortgage debt.

" It may perhaps be helpful if I indicate the considerations " and conditions which, in my opinion, can fairly be regarded " and applied in ordinary cases, by which I mean cases in " which the relationship of mortgagor and mortgagee subsists " without any complication arising out of any other contractual " relations between the parties, or because the position of either " party is exceptional.

" In such cases the following furnish good working rules:

" (*a*) If the security is sufficient, and if the covenants and " conditions of the mortgage deed—other than the covenant to " repay the principal moneys—have been performed and " observed the mortgagor ought to be given a reasonable time " within which to find the money to pay the debt.

"(*b*) The reasonable time so given may be extended if, " in cases where the interest reserved is less than 5 per-cent, " the mortgagor be willing to pay interest at the rate of " 5 per-cent, and still further extended if, in addition, he is " willing to pay over to the mortgagee on account of principal " the difference between the net rents received from the " mortgaged property and the interest at 5 per-cent.

"(*c*) In a like case, if the mortgagor is in occupation and the " proper occupation rent is in excess of 5 per-cent interest on " the debt he ought to attorn tenant to the mortgagee at the " proper occupation rent, and pay to the mortagee interest at " 5 per-cent and the excess of such rent over the 5 per-cent " interest on account of capital.

"(*d*) If the security is insufficient but interest has been " paid up to date, and the other covenants and conditions " in the mortgage deed have been performed and observed the " mortgagee should not be exposed to further loss, and time " should be given to the mortgagor on the terms that interest " at 5 per-cent is paid, and that the mortgagee may renew the " application if and when any further depreciation in the security " takes place.

(*e*) If the security is insufficient and there are arrears of " interest, but not to a serious amount, the mortgagor should " be required to clear off such arrears in addition to complying " with the terms set out in (*d*).

(*f*) "If in either of the last two cases the rent of the " mortgaged property or a proper occupation rent therefor " exceeds the interest the mortgagee should be allowed, if he so " desires, to appoint a receiver, but in the event of his so doing, " in a case falling under (*e*), the stipulation requiring the mort- " gagor to pay off the arrears of interest may have to be modified " by giving him a reasonable time within which to clear them off.

"(*g*) In cases where there are substantial arrears of interest, " or where the covenants to keep down ground rents and other " outgoings, or to keep in repair or insure have been broken, the " mortgagee should not be restrained from exercising his " powers, in the absence of very special circumstances, unless " the mortgagor is prepared to make good all such breaches " forthwith and to continue under conditions similar to those " enumerated under (*d*), (*e*), and (*f*).

"Now it is clear that in this case if there was no war the " mortgagors would have had to provide for repayment by a

" transfer. It was argued that they had not brought themselves " within the protection of the section. But I am satisfied, on " the evidence and having regard to the general conditions of " which I am bound to take judicial notice, that it is almost " impossible to obtain the necessary advance of money by any " transfer of the mortgage except on terms which would leave " the borrower in a position which is many times worse than " that which he occupies in existing circumstances. I am quite " satisfied that they have established a case within the section. " The mortgagors have brought themselves within rule (*a*) " mentioned above, and they ought to be given a reasonable " time within which to find the necessary money.

" On the mortgagors undertaking to increase the interest to " 5 per-cent and to repay out of principal £500 per annum in " October, I defer the operation of the mortgagee's remedies for " one year from to-day, if the war so long continues, with liberty " to apply in the event of the mortgagors failing to keep their " undertaking. The costs must be added to the security."

Stamp duty on mortgages for unlimited amounts.

I am indebted to Mr. G. J. Lidstone for a copy of correspondence with the Board of Inland Revenue on the subject of stamp duty on mortgages for unlimited amounts.

The previous practice of the Inland Revenue Authorities is dealt with by Mr. Barrand in "Further Notes on Some Legal Aspects of Life Assurance Practice", *J.I.A.*, vol. xli, pp. 210, 211, and the extract given below from a letter from them dated 7 January 1918, indicates an alteration in their attitude.

In the case in question a mortgage, dated 1 July 1884, for an unlimited amount was stamped £6, that is to cover advances up to £4,800, and the security consisted of life policies in several offices. By assignment, dated 30 January 1885, the mortgagees sold all the policies to third parties for a total consideration of £1,524—which it will be observed was well within the amount covered by the stamp duty—but the policies were kept up by the purchasers until 1917 when the life assured died, and the total amount, which has recently become payable to the assignees, is considerably in excess of £4,800, the amount covered by the stamp. In the first instance the Inland Revenue Authorities ruled that the offices collectively could not pay a larger total amount than £4,800 without requiring the mortgage to be further stamped.

Further correspondence ensued and finally the Board wrote:

"Where a mortgagee exercised his powers of sale under a "mortgage and sold the policies comprised therein the Board "will not hold insurance companies responsible for penalty "under Section 118 of the Stamp Act of 1891 if they pay the "full amount of the policy claims, even though that amount may "exceed the amount which the mortgage is stamped to cover, "provided that at the date of the assignment to the purchaser "from the mortgagee the total of the surrender values of the "policies did not exceed the amount which the mortgage is "stamped to cover. In the present case it appears that the "mortgagee sold the policies under the powers of sale conferred "by the mortgage, and that at the date of sale the surrender "values of the policies amounted to less than £4,800, the "amount on which the mortgage is stamped, and the Board "will accordingly offer no objection so far as they are concerned "to the payment of the policy moneys."

Policy issued to cover funeral expenses. Conflict of claim.

The unreported case of *Da Costa* v. *Prudential Assurance Company* will probably be of interest to industrial assurance officials by reason of the decision of the Court of Appeal upon the meaning of Section 36, sub-section (2) of the Assurance Companies Act 1909.

The facts are as follows:

Two policies of assurance for a total sum of £21 were taken out in 1900 and 1905 respectively upon the life of one Mary Nicholl. They were effected by her son Joseph Sylvester Nicholl, who paid premiums upon them. Mary Nicholl died on 25 December 1916, and the sum assured was paid by the Company to Joseph Sylvester Nicholl, who delivered up the policies. An action was brought by the executrix of Mary Nicholl, a Mrs. Da Costa, who obtained judgment in the Liverpool Court of Passage for £21 against the Company. Section 36, sub-section (2) of the Assurance Companies Act, 1909, provides:

"No policy effected before the passing of this Act with a "collecting society or industrial assurance company shall be "deemed to be void by reason only that the person effecting "the policy had not, at the time the policy was effected, an "insurable interest in the life of the person assured, or that "the name of the person interested, or for whose benefit or

" on whose account the policy was effected, was not inserted " in the policy, or that the insurance was not one authorized " by the Acts relating to friendly societies, if the policy was " effected by or on account of a person who had at the time a " *bonâ fide* expectation that he would incur expenses in con- " nection with the death or funeral of the assured, and if the " sum assured is not unreasonable for the purpose of covering " those expenses, and any such policy shall enure for the " benefit of the person for whose benefit it was effected or " his assigns."

The Company having appealed from the judgment of the Liverpool Court, the appeal was heard before Swinfen Eady, L.J., Bankes, L.J., and Eve, J., and judgment given for the appellants.

In the course of his judgment Lord Justice Swinfen Eady said :

" The learned Judge in the Court below has found that Mrs. " Nicholl's son took out the policies in his mother's name in " relation to any expenses in connection with her death he " might be called upon to bear. I do not think there is any " doubt about that. He also finds that the two sums for which " the policies were effected were not unreasonable.

" Then it appears that after the death of Mary Nicholl, " which took place on the 25 December 1916, the son made " good his title to both policies to the satisfaction of the " Prudential Assurance Company, and they paid him the " amount and took from him a receipt, and he delivered up the " policies. The language of the policies is that upon payment " of the premiums and on satisfactory proof of the death of the " assured the company will pay the amount assured to the " executors or administrators. Then that is followed by a " proviso : Provided always that the production by the company " of a receipt for any sum payable hereunder, signed by any person " being either an executor or administrator or the husband " or wife or a relation by blood or connection by marriage of " the assured shall be a discharge to the company for the same, " and shall be final and conclusive evidence to all intents and " purposes that such sum has been duly paid and received by " the person or persons lawfully and rightfully entitled to the " same, and that all claims and demands whatsoever against " the company in respect of this policy have been fully satisfied. " What has happened is that the Prudential Assurance " Company had produced receipts signed by the son, who is, of

"course, a relation by blood, and they say that is a discharge "to the company.

"Of course if that defence is well founded, it is a good "discharge. But it does not rest there, because upon the facts, "as the learned Judge has found them, the policy enures to the "benefit of the son, and if the plaintiff were to recover she "would only recover as trustee for the son, and her duty then "would be to pay the sum that she recovered over to the son. "But Joseph Sylvester Nicholl has already been paid direct by "the Insurance Company the whole amount, and where money "in that way has got home to the *cestui que trust* it is not "competent for an executor or trustee to sue again for the "money with the intent that he or she shall pay it over again "to the *cestui que trust.*

"Then it was urged that the present case was not within the "section, because the name of Joseph Sylvester Nicholl does not "appear in the policy itself. But the section is intended to meet "a case of that kind. It says that 'No policy effected before "the passing of this Act'—and both these policies fulfilled this "condition—'shall be deemed to be void by reason only that "'the person effecting the policy had not, at the time the policy "'was effected, an insurable interest in the life of the person "'assured, or that the name of the person interested, or for "'whose benefit or on whose account the policy was effected, "'was not inserted in the policy.' The section continues 'any "'such policy shall enure for the benefit of the person for whose "'benefit it was effected', that is to say, although the name was "not inserted in the policy, and although the policy itself is "silent on the question, nevertheless it shall enure for the "benefit of the person for whose benefit it was effected.

"Therefore, in my opinion, the appeal should be allowed, "and judgment entered for the defendants."

Lord Justice Bankes and Mr. Justice Eve both agreed.

Annuity Reserves and Rates in the United States.

On 8 November last Mr. D. P. Fackler wrote as follows to Mr. J. D. Watson, Joint Hon. Secretary of the Institute:

"As Corresponding Member of the Institute for the United "States, it appears to be my duty to inform you and the Institute "of the following facts regarding Annuties:

"The Convention of the Insurance Commissioners of the several "States at their Meeting in the autumn of 1916 adopted a resolution "to the effect that in computing the legal reserves to be held for "life annuities the rate of interest assumed should be one-half of "1 per cent. higher than the legal rate prescribed for the computa- "tion of the reserves for life insurances. The resolution stated the "reasons for this recommendation, which was made on the assump- "tion that the annuity reserves would be computed upon a table "fairly representing annuity experience.

"In consequence of the above action, the Legislature of the "State of New York early in this year (1917) amended the laws "governing the proper legal reserves to be held for insurances and "annuities, so that the legal rate to be used in the computation of "annuity reserves should be 4 per cent., whilst that to be used for "life insurances is $3\frac{1}{2}$ per-cent.

"Many other States will, doubtless, follow New York in the "above action when their Legislatures meet this winter. Several "insurance companies have already reduced their premiums for "annuities in consequence of the above change in the New York "Law, and it is likely that many others will soon do the same.

"Your Corresponding Member has been much interested in the "general subject of annuities, and presented articles thereon at the "Meetings of the Actuarial Society in May 1914, and in October "1915, which may be found in the Society's *Transactions*, and may "have, in some measure at least, contributed to bring about the "above legislation."

The United States Government Life Assurance Scheme for Officers, Men and Nurses on Active Service.

[We are indebted to Mr. W. A. Hutcheson, F.I.A., F.F.A. F.A.S., for a copy of the Act of the United States Legislature by which provision was made for the assurance of officers and men—and the nursing staff (male and female)—of the United States Army and Navy by the Treasury War Risk Insurance Bureau. We extract the following sections from Article IV (Insurance).—Eds. *J.I.A.*] :

Sec. 400. That in order to give to every commissioned officer and enlisted man and to every member of the Army Nurse Corps (female) and of the Navy Nurse Corps (female) when employed in active service under the War Department or Navy Department greater protection for themselves and their dependents than is provided in Article III,* the United States, upon application to the bureau and without medical examination, shall grant insurance

* This article provides (without payment of any premium) for compensation in the event of disablement, and for benefits to dependent relatives in the event f death.—Eds. *J.I.A.*

against the death or total permanent disability of any such person in any multiple of $500, and not less than $1,000 or more than $10,000, upon the payment of the premiums as hereinafter provided.

SEC. 401. That such insurance must be applied for within one hundred and twenty days after enlistment or after entrance into or employment in the active service and before discharge or resignation, except that those persons who are in the active war service at the time of the publication of the terms and conditions of such contract of insurance may apply at any time within one hundred and twenty days thereafter and while in such service. Any person in the active service on or after the sixth day of April, nineteen hundred and seventeen, who, while in such service and before the expiration of one hundred and twenty days from and after such publication, becomes or has become totally and permanently disabled or dies, or has died, without having applied for insurance, shall be deemed to have applied for and to have been granted insurance, payable to such person during his life in monthly instalments of $25 each. If he shall die either before he shall have received any of such monthly instalments or before he shall have received two hundred and forty of such monthly instalments, then $25 per month shall be paid to his wife from the time of his death and during her widowhood, or to his child, or widowed mother if and while they survive him : Provided, however, That not more than two hundred and forty of such monthly instalments, including those received by such person during his total and permanent disability, shall be so paid ; and in that event the amount of the monthly instalments shall be apportioned between them as may be provided by regulations.

SEC. 402. That the director, subject to the general direction of the Secretary of the Treasury, shall promptly determine upon and publish the full and exact terms and conditions of such contract of insurance. The insurance shall not be assignable, and shall not be subject to the claims of creditors of the insured or of the beneficiary. It shall be payable only to a spouse, child, grandchild, parent, brother or sister, and also during total and permanent disability to the injured person, or to any or all of them. The insurance shall be payable in two hundred and forty equal monthly instalments. Provisions for maturity at certain ages, for continuous instalments during the life of the insured or beneficiaries, or both, for cash, loan, paid-up and extended values, dividends from gains and savings, and such other provisions for the protection and advantage of and for alternative benefits to the insured and the beneficiaries as may be found to be reasonable and practicable, may be provided for in the contract of insurance, or from time to time by regulations. All calculations shall be based upon the American Experience Table of Mortality, and interest at three and one-half per centum per annum, except that no deduction shall be made for continuous instalments during the life of the insured in case his total and permanent disability continues more than two hundred and forty months. Subject to regulations, the insured shall at all times have the right to change the beneficiary or beneficiaries of such insurance without the consent

of such beneficiary or beneficiaries, but only within the classes herein provided. If no beneficiary within the permitted class be designated by the insured, either in his lifetime or by his last will and testament, or if the designated beneficiary does not survive the insured, the insurance shall be payable to such person or persons, within the permitted class of beneficiaries as would under the laws of the State of the residence of the insured, be entitled to his personal property in case of intestacy. If no such person survive the insured, then there shall be paid to the estate of the insured an amount equal to the reserve value, if any, of the insurance at the time of his death, calculated on the basis of the American Experience Table of Mortality and three and one-half per centum interest in full of all obligations under the contract of insurance.

Sec. 403. That the United States shall bear the expenses of administration and the excess mortality and disability cost resulting from the hazards of war. The premium rates shall be the net rates based upon the American Experience Tables of Mortality and interest at three and one-half per centum per annum.

Sec. 404. That during the period of war and thereafter until converted the insurance shall be term insurance for successive terms of one year each. Not later than five years after the date of the termination of the war as declared by proclamation of the President of the United States, the term insurance shall be converted, without medical examination, into such form or forms of insurance as may be prescribed by regulations and as the insured may request. Regulations shall provide for the right to convert into ordinary life, twenty payment life, endowment maturing at age sixty-two and into other usual forms of insurance and shall prescribe the time and method of payment of the premiums thereon, but payments of premiums in advance shall not be required for periods of more than one month each and may be deducted from the pay or deposit of the insured or be otherwise made at his election.

REVIEWS.

The Mathematical Theory of Population, of its Character and Fluctuations, and of the Factors which influence them. By G. H. Knibbs, C.M.G., F.S.S., F.R.A.S., *&c.*

[Pp. xvi + 466. Melbourne: McCarron, Bird & Co.]

This lengthy appendix to the census of the Commonwealth of Australia discusses the use of algebraic or trigonometric functions in interpreting population statistics, and applies the methods to the Australian census. After an introductory chapter Mr. Knibbs proceeds to examine the way in which population fluctuates, and although he gives some expressions that may take a more general form, he seems to think that the normal basis of change of population is of the nature of a geometrical progression. He implies that the

geometrical form may not be realized, but harks back to it, and must, we think, feel that this is the proper starting point. There is much to be said for this assumption on *a priori* argument, but such assumptions are apt to be upset by a "disagreeable little fact." When, however, Mr. Knibbs finds that a variation in the rate of change is necessary he comes much nearer to the facts of the case, and he suggests the use of $1+\eta t^{m+nt}$ as an expression for dealing with the rate of change of population; the constants being obtained from three observations "at any suitable intervals."

In a subsequent chapter Mr. Knibbs apparently leaves the question of curves suitable for rates of increase and turns to fitting the facts themselves. He refers to the Pearson type curves, and after saying "although the type curves . . . fulfil their general "purpose fairly well, experience shews that their '*fitting pouer*' "is somewhat limited", he gives what he calls a "flexible curve" of the form $Ax^m e^{nx^p}$. This curve does not lend itself to fitting by moments or by a systematic method, and Mr. Knibbs says we must take several series of ordinates; each set "will give a value "for p . . . . and a mean (geometric, arithmetic or other) can be "taken." In spite of Mr. Knibbs's experience of the Pearson-curves there is surely ample evidence that their fitting power has a wide range, and it is hard to see why we should desert a proved group of curves which can be fitted systematically, and has some theoretical justification, and use instead an arbitrary curve, the constants of which must be found by the method indicated above. The Pearson-curves may have to give place in the future to some still more useful group, but at the present time they stand out as the most practical attempt to find curves suitable for the description of frequency distributions, and except when Mr. Knibbs's curve takes the form of the Pearson Types III and V, we do not think he will find other people anxious to use it.

Mr. Knibbs discusses the result of certain projections of the normal curve—a method to which Professor Edgeworth has given much attention—but he seems to us to project the normal curve on to surfaces chosen arbitrarily, and if we make the form of the surface on which we project sufficiently complex we can, doubtless, obtain a wide variety of forms, most of which will, however, be useless for curve fitting in practice. This part of the work, in common with many other parts, gives the impression of notes of ideas that have occurred to the author rather than considered results of research.

The next sections discuss group values, integration for statistical purposes and smoothing in population statistics, and then after a preliminary chapter we come to the arithmetical part of the work, although several pages dealing with the summation method of graduation and various other incidental or algebraic work are interspersed. Some of the tables give interesting information, and the tables dealing with errors in age and with marriages, as well as some of the tables on fertility, &c., are specially of value.

The book is well printed and contains some excellent diagrams, but it is far too long, and the excess of matter leaves the reader with

a feeling of bewilderment. If the more important investigations and tables could be separately printed in a volume of greatly reduced size, they would be more likely to receive the study they deserve.

W. P. E.

Du Calcul dans les Jeux de Hasard. Rédigé par D. J. Korteweg. *Extrait des œuvres complètes de Christiaan Huygens: Tome* XIV.

The direct contribution of Huygens to actuarial science does not seem to have been of great importance, although it is of some interest historically as the earliest recorded application of probabilities to life contingencies.

In the correspondence of the brothers Christiaan and Lodewijk (*J.I.A.*, vol. xxxiv, p. 387) on the subject of the calculation of expectations of life from Graunt's Mortality Table, Lodewijk gave the more practical solution of the problem—a computation, rough certainly, but based on correct principles, of what we now call the expectation of life—if Christiaan had the better of the argument.* It is true that Christiaan, with his singular sincerity and indefatigability in clearing up difficulties, discovered in the end that they were both right ("je trouve que nous avons tous deux raison en prenant la chose en different sens") and that Lodewijk's expectation of life and his own *Vie Probable* were two distinct things. But at the outset, influenced probably by the fact that the infant science of probabilities had up to then been applied almost exclusively to games, he seems to have regarded the problem as of the nature of a bet ("Qui gagerait qu'un enfant de 6 ans vivra jusqu'à 26"—Lodewijk having computed the expectation at age 6 at 20 years 10 months—"peut mettre 25 contre 39, "puisque de 64 enfans de 6 ans, il y en a 25 qui parvienent à l'aage "de 26 ans contre 39 qui meurent au dessous"), and it was only towards the end of the correspondence that he realized that it had a practical connection with the important subject of life annuities ("Le premier", *i.e.*, Lodewijk's expectation, "est pour regler les rentes à vie, et l'autre pour les gageures".) With reference to this correspondence Dr. Korteweg suggests that the "courbe de vie", given by Huygens in his concluding letter, is the first example of the graphic representation of mortality; but the curve in question appears to be an interpolation curve for the purpose of finding the values of the expectation of life at intermediate ages from the computed values for ages 0, 6, 16, 26, &c., rather than a mortality curve as ordinarily understood.

The only other recorded instance of Huygens having interested himself in the actuarial applications of probabilities is in connection with Hudde's investigations (*Memoires pour servir, &c.*, pp. 76, 82), in which however it would not appear from Hudde's letters that he took any more active part than that of sympathetic appreciation.

* This correspondence was published in vol. vi of the *Oeuvres Complètes*, and was reproduced shortly afterwards, by permission of the Editors, in the *Memoires pour servir*.

But if Huygens contributed little directly to actuarial science, the value of his indirect contribution can hardly be over-estimated, for it is to him that we are indebted for the first clear statement of the elementary principles of probability. Although Pascal and Fermat had already proposed and solved some of the problems with which Huygens dealt, their methods had not been disclosed at the time when his attention was directed to the subject ("ces savants, "quoiqu'ils se missent à l'épreuve l'un l'autre en se proposant "beaucoup de questions difficile à résoudre, ont cependant caché "leurs méthodes"), and it was left to Huygens to construct from first principles a systematic method of procedure. The result was his classical "De Ratiociniis in Ludo Aleæ"—to give the tract which forms the subject of this notice its better known Latin title*—a model of clearness and lucidity, which "continued to form the "best account of the subject until it was superseded by the more "elaborate works of James Bernouilli, Montmort and De Moivre" (*History of the Theory of Probability*, p. 25). In Bernouilli's *Ars Conjectandi*, published more than half a century later, an annotated reprint of the "De Ratiociniis" occupies the first part, and in many later works the influence which it exercised on the development of the subject of probability may be traced. English translations from the original Dutch and Latin appeared in 1692 and 1714, and in recent times a French translation has been published in the *Memoires pour servir*, and an English translation (by Mr. E. W. Scott) in the Transactions of the Actuarial Society of America (Vol. IV). But it has appropriately, if somewhat exceptionally, been reserved to Huygens' country to render due honour to the tract, and this has been most handsomely done in the definitive edition of his works which has been in course of publication by the Société Hollandaise des Sciences during the last quarter of a century. The extract which has now been separately issued under the title given at the head of this notice renders available to students of the early history of probability practically all that Huygens wrote on the subject, including, in addition to the "De Ratiociniis," nine appendixes containing various other researches. The value and interest of the publication are much enhanced by the Editor's historical introduction and mathematical commentary—and, it may be added, by such luxuries, unusual in these days, as large paper, wide margins, and an admirable fount of type.

The problems discussed by Huygens are for the most part special and comparatively simple cases of problems which have since become historical, such as the Problem of Points, the Problem of Dice, and the great Duration of Play Problem. They have long since been generalized, and have been solved in their general form by more powerful methods of analysis than were available in the early days of probabilities. Nevertheless the

* A copy of the original Latin edition of Van Schooten's *Exercitationes Mathematicæ* (1657), in which the *De Ratiociniis* was first published, has recently been added to the Institute Library.

solutions given by Huygens are instructive on account of the simplicity of his analytical apparatus and the lucidity of his reasoning. Starting from the hypothesis that the chance that a player has of winning in a game has a value such that if he possessed that value he could secure the same chance in a game in which he had an equal chance of winning or losing, he establishes three elementary propositions of the category represented by the general principle that, if A's chances of winning a_1, a_2, &c., are p_1, p_2, &c., respectively, the value of his expectation is $\Sigma pa/\Sigma p$. This forms the basis of his argument. His one method of solving the various problems proposed by Pascal and others, or devised by himself, consists, as the Editor of the present reprint points out, "dans une application continuelle, répétée autant de fois que le "problème l'exige, de ces propositions." It is remarkable how much he achieves by such simple means, and if the method becomes laborious and its repetition monotonous as the problems become more complicated, it could, in some cases, be simply used in association with the method of induction, as, for example, in the Problem of Points, or in the Duration of Play Problem when the game is supposed to continue until one player has nothing left.

We may add, in conclusion, that at the end of the Editor's introduction will be found an interesting discussion of the "dédale des problèmes" exchanged by Huygens and Hudde on the advantage or disadvantage of playing first in certain games of chance. In a game, for example, of the pitch and toss variety in which two players toss in turn and each puts one in the pool when a head turns up and takes one out when tail turns up, and it is a question of the disadvantage of playing first (when there is nothing to take out) or generally of how the pool should be divided at any time, both Huygens and Hudde assumed that the expectations of the two players must be together equal to the amount in the pool. But Dr. Korteweg points out that the game may be prolonged indefinitely, and that in such cases there is a third expectation which he proposes to call "le part de diable." The devil's share in the case in question is the whole amount in the pool, the players having definitely lost—in the absence of any arrangement for ending the game—all that they have put in.

Interest and Bond-values. By M. A. MACKENZIE, M.A., F.I.A., A.A.S. *Second Edition. Revised and Enlarged.*

[Pp. 107 and Tables pp. x. University of Toronto. Price $2.]

(London: C. & E. Layton. 8*s*. 6*d*. net.)

WE welcome a second edition of Prof. Mackenzie's useful little text-book on the theory of interest and its practical application to the valuation of bonds and other securities.

Some alterations have been made in the earlier chapters, a certain amount of new matter has been inserted (we notice, for

example, an interesting discussion on p. 72 of the question whether it is better from the borrower's point of view to issue a loan at a premium or at a discount when repayment has to be provided for by a sinking-fund accumulated at a rate lower than the yield on the loan), and the exercises at the end have been supplemented by a few examination papers, which may assist students to test their knowledge of the subject.

A feature of the book which will commend it to those who find it easier to understand principles and methods when illustrated arithmetically than when expressed in symbols is the prominence given throughout to practical numerical examples. In valuing redeemable securities Prof. Mackenzie shows how Bond-value Tables may be used in some cases in which the conditions are not exactly those assumed in the construction of the tables, but we gather that he considers it simpler in such cases—and in fact safer in all cases unless the Bond-value Tables have been checked and the assumptions on which they have been constructed are clearly stated—to use ordinary interest and annuity-tables. The question of *income-tax*, which is now in the United Kingdom (and may in course of time become in America) such a very material factor in the valuation of, and the calculation of the yield on, redeemable securities, is not discussed.

In the probable event of future editions of the book being required, the table on p. 32—showing on a simple interest basis equivalent yearly, half-yearly and quarterly payments at various rates of interest—might perhaps be reconsidered. Although the simple interest equivalents may be near enough for practical purposes, a table of this kind, in which some of the tabulated amounts are correct (to the number of places of decimals retained), and others are only approximate, and its employment in many of the numerical examples, seems to us to make it harder for the student to understand clearly the important distinction between effective and nominal rates. On p. 18 it is stated that "if the "interest-rate be 4 per-cent compounded half-yearly, the payments "of \$100 at the end of each quarter must be regarded as payments "of \$201 at the end of each half-year; while payments of \$100 at "the end of each year must be regarded as payments of \$49·505 at "the end of each half-year." If the equivalent payment is required to three places it would seem no less important in the former case than in the latter that it should be calculated accurately. The general statement that "a payment of 1 at the end of each year is "equivalent, at a rate of i per year, to a payment of x at the end "of each half-year where $x+x\left(1+\frac{i}{2}\right)=1$" is equally open to question. We can quite understand that in a practical book it may be desirable to avoid the complications of $i/j_{(p)}$, but since—as Prof. Mackenzie states on p. 44—"the investment yield compounded "half-yearly has become almost an institution", the practical course would appear to be to deal only with a half-yearly effective rate, and to work out everything correctly on that basis.

Another point that may be mentioned is the retention of cents, *i.e.*, two places of decimals, whether they are justified or not, in the results of most of the numerical examples. Thus, on p. 43, the value to yield $4\frac{1}{2}$ per-cent, compounded half-yearly, of a \$10,000 bond due in 25 years, and bearing quarterly coupons at 4 per-cent, is worked out at \$9287·55, when in the calculation the equivalent half-quarterly coupon is taken as \$201·12 and multiplied by $a_{\overline{50}|}$ at $2\frac{1}{4}$ per-cent. And on p. 66 the value to pay $4\frac{1}{2}$ per-cent compounded half-yearly of a \$50,000 25-year 5 per-cent annuity bond with quarterly payments is given as \$52,726·63, this being the product of $a_{\overline{50}|}$ (at $2\frac{1}{4}$ per-cent) and an approximation (to two places on a simple interest basis) to the equivalent half-yearly payment of the annuity. Clearly the cents cannot be reliable in either case. In some cases, where the factors to be multiplied are exact, Prof. Mackenzie obtains correct results by using interest functions to more places than are given in the tables at the end of the book or are to be found in most practical tables. It would in our view be more instructive to base the examples on the tabulated values, and to let the student see what degree of accuracy can be obtained with the means available. The resulting values would in most cases be practical—cents being quite meaningless in connection with tens of thousands of dollars. In the few cases in which, for comparison of methods or for making a loan-schedule work out neatly, greater accuracy is necessary, the values of the functions to the requisite number of places could be specially stated.

CORRESPONDENCE.

ANNUITY-VALUES ON MAKEHAM'S HYPOTHESIS.

To the Editors of the Journal of the Institute of Actuaries.

Sirs,—The review (*J.I.A.*, vol. l, p. 320) of my paper "Su una "relazione fra l'annualità vitalizia di gruppo e l'annualità semplice, "nell'ipotesi di Makeham," suggests the following observations:

(1). The reader might suppose that I had repeated—although without being aware of the fact—McClintock's analysis in "On the "computation of Annuities on Mr. Makeham's Hypothesis" (*J.I.A.*, vol. xviii, p. 242). But that is not the case either in form or in substance.

By means of the B-function—which McClintock did not consider—I endeavour to establish whether, and if so on what conditions, continuous annuity-values for m joint lives can be obtained by the formula

$$\bar{a}_{x_1x_2\,\cdots\,x_m}=\frac{1}{p_m\log c}\left[\mathrm{F}(1-p_m,\,q_m)-c^{q_m}q_m^{p_m}\Gamma(1-p_m)\right]\quad.\quad(1)$$

where F is a hypergeometrical function and Γ the Γ-function.

Neither van der Belt, to whom the corresponding formula for $m=1$ is attributed (Enc. des Sciences Mathematiques, T.I., vol. 4, p. 531), nor McClintock, whose priority ought to be recognized—nor, so far as I know, any other writer—ever suggested the generalization and conditions established by me.

(2). Formula (1) is valid so long as the function F is convergent, and so also is the formula

$$a_{xy} = \mathrm{K} a_x \ . \ . \ . \ (2)$$

which can be readily deduced from (1). The coefficient K depends on the functions F and Γ, and the calculation of its numerical value can be carried to any degree of approximation.

In the case of the *Text-Book* $3\frac{1}{2}$ per-cent Table, formulas (1) and (2) are applicable so long as the number of lives m does not exceed 9. If $m=2$ the formulas are applicable so long as w does not exceed 67, where $2c^w = c^x + c^y$; and if $m=3$ they are applicable so long as w does not exceed 63, where $3c^w = c^x + c^y + c^z$. The resulting coefficient K is always positive.

It follows that the reviewer's statement that the formula "seems to be inapplicable (the numerator becoming negative) to "such a practical case as the evaluation of $a_{60.60.60}$", cannot refer to formulas (1) and (2), and I do not know how it can refer to formula IV′, because the numerator of this formula is not negative under the stated conditions, and I had anticipated the reviewer by stating that the formula is available for values of q_m "abbastanza piccoli" (rather small) and consequently not for greater values of m and older ages.

(3). When commutation-tables for a particular mortality-table are not available, and it is not convenient for any reason to undertake the rather laborious work of tabulation, it seems undeniable that formula (1) with the indicated limitations, and the passage from $\bar{a}_x$ to a_x, will suffice for all requirements. Further, if one has a single-life commutation-table or the single-life annuity-values for all ages, and it is inconvenient to construct commutation-tables for two or more lives of equal ages, formula (2)—which admits, by the simple process of calculating the coefficient K, of the passage to joint-life annuities—is not to be despised.

The coefficient K can of course be represented by several approximate expressions. For instance, I gave a first approximation applicable to the RF $3\frac{1}{2}$ per-cent Table. The corresponding approximation for the *Text-book* $3\frac{1}{2}$ per-cent Table—to pass from a_x to a_{xx}—would be

$$\mathrm{K} = \frac{\cdot 12 + \cdot 49f - \cdot 31(1+2f)f^{\cdot 51}}{\cdot 14 + \cdot 25f - \cdot 22(1+f)f^{\cdot 45}}$$

where $$f = \cdot 00105 + \cdot 000096x + \cdot 0000044x^2$$

and it can hardly be said that this involves "a somewhat laborious calculation."

(4) Finally, the writer of the review appears to give the preference to the formula $a_{xy} = a'_w$. But for the application of this

formula it is necessary to have annuity-values for all ages and for several rates of interest, or to seek the assistance of other more or less approximate formulas. Therefore, generally, one must be content with an approximation which has not been proved to be satisfactory in all cases in its results. Formula (2) on the other hand is always applicable when the single-life annuity-value is given at the same rate of interest; and formula (1) is applicable in every case without any preliminary tabulation.

I am, Sirs, &c.,

F. INSOLERA.

R. Ist. Sup. di Commercio, Turin,
23 *January* 1918.

[We are glad to publish Prof. Insolera's letter, but we do not think that there was anything in our review to suggest that his analysis was the same as McClintock's. The reviewer's statement that Prof. Insolera had used McClintock's method (*i.e.*, the method of evaluating the integral for $\bar{a}$ in an infinite series) for the purpose of obtaining the ratio of $a_{xyz}\ldots$ to a_x appears to be in accordance with the facts. McClintock did not restrict his investigation to the case of a single life; he indicated that the method could be applied, by a simple modification, to m lives, and his formula, modified accordingly, is identical in substance with Prof. Insolera's generalised formula (1). With regard to the conditions of applicability of the formula, the limitations imposed by Prof. Insolera do not appear to be necessary. Although the hypergeometrical series, in its general form, is divergent if x is >1, the special type of hypergeometrical series entering into McClintock's and Prof. Insolera's formula is convergent for all values of Σc^x. It follows that the formula is valid for lives of any ages, and it may be extended to any number of lives by further integrations by parts. The objection to the formula is not that it is subject to any limitations in theory, but that it involves an impracticable amount of calculation—owing to the slow convergency of the series—except for young lives. We are indebted to Mr. G. J. Lidstone for the information that when $\Sigma c^x \log_e 1/g$ is large a good result can be obtained by means of Schlömilch's series for the incomplete Γ-function (*see* Bromwich's "*Introduction to the Theory of Infinite Series*"). There would seem to remain, however, a considerable interval between the age at which McClintock's formula ceases to be

of practical utility and that at which the Schlömilch series becomes applicable.

The statement quoted by Prof. Insolera in (2) refers to his formula IV′. This formula is given for $a_{x_1x_2\ldots x_m}$ without any explicit limitation of its applicability, and the words at the beginning of the investigation "quando si abbia da fare con gruppi di pochi elementi, cosi che q_m sia abbastanza piccolo" (when one has to do with combinations of a few lives so that q_m is sufficiently small) would not, we think, lead the ordinary reader to suppose that the formula does not apply to the calculation of a joint-life annuity on three lives of 60.

With regard to the approximation to K given in (3) it should be borne in mind that the expression is derived from formula IV′ and is of limited application. It appears to give $a_{65.65}=6{\cdot}375$, the true value being 5·486. The approximate formula $a_{xy}=a_w+\log_e s(Ia)_w$, gives (without using tables at more than one rate of interest) the correct result 5·486. —Eds. *J.I.A.*].

MORTALITY AMONG NEUTRALS IN WAR-TIME.

To the Editors of the Journal of the Institute of Actuaries.

Dear Sirs,—Those Members of the Institute who read Professor Hersch's paper "La Mortalité chez les Neutres en Temps de Guerre", reviewed in *J.I.A.*, vol. l, p. 72, will remember that in this paper the author endeavoured to answer the question: "Which classes "of a population are most seriously affected by the indirect effect "of a War?"

The method adopted by the author was to consider the increase of mortality due to a War as the *absolute difference* between the mortality experienced in a time of War and the normal mortality of a time of peace, and to compare the results thus obtained for the different age groups. The method was, in fact, equivalent to a comparison of $q_{ng}-q_n$ age-group by age-group, where q_{ng} represents the mortality from all causes, including the indirect effect of a War, and q_n the normal mortality.

The same subject was dealt with by Mr. J. W. Nixon in his paper "War and National Vital Statistics with Special Reference "to the Franco-Prussian War"—Journal of the Royal Statistical Society, vol. lxxix, part 4. In this paper the author contended that the proper method of comparison was to compare, not the absolute, but the percentage increase in mortality, *i.e.*, not $q_{ng}-q_n$, but

$$\frac{q_{ng}-q_n}{q_n}$$

By adopting this method Mr. Nixon arrives at the following conclusions:

"It will be seen that the highest age-groups generally show the "smallest increases in mortality, the highest increases being shown "at periods of young and middle life. This, I think, is contrary "to expectation. It in also contrary to certain definite conclusions "based on the same figures recently made in a Swiss publication. "This pamphlet states that the effects of War on mortality are the "greatest at the two extremes of life, and are lowest at the ages "10 to 14. . . . . . The method by which these conclusions are "reached is fallacious."

Mr. Nixon then discusses Prof. Hersch's method in a foot note:

"As showing the incidence of mortality on the two sexes the "author seems quite sound, but in dealing with the incidence of "mortality at different ages of life his analysis is statistically quite "unsound. . . . . . The author takes the mean number of deaths "in the years 1870–71, and finds the excess in these two years "over the year 1869 per 1,000 of the population living at each age-"group. He thus obtains a curve which reproduces in general "outline the usual U-shaped mortality curve for ages, namely, a "very high death-rate at the two extremes of life, and a low death-"rate for the intervening ages. As persons over 65 had a high "death-rate in 1869, and also a high death-rate in the years "1870–71, the difference between these two is likely to be much "higher than the difference between the death-rates of a young "age-group, *e.g.*, 15–20. By using this method of absolute increase "in death-rates the Author, in effect, says that a rise in the death-"rate from (say) 10 to 15 per 1,000—*i.e.*, a rise of 5 per 1,000—"is much more serious than a rise in the death-rate from (say) "3 to 6 per 1,000—*i.e.*, one of 3 per 1,000. The former, however, "is a rise of 50 per cent, the later of 100 per-cent, and these "figures are the ones which should be compared."

These remarks of Mr. Nixon's appear to have aroused considerable interest in Italy, and have given rise to two papers, the first of which, by Corrado Gini, appeared in the September–December 1916 number of the "Rivista Italiana di Sociologia", while the other by F. P. Cantelli, entitled "Sull 'Aumento di "Mortalità dovuto alla guerra. Riflessioni critiche di metodologia "statistica", originally appeared in the November 1917 number of the "Giornale degli Economisti e Rivista di Statistica", and has now been published in pamphlet form. Both of these writers take exception to the criterion proposed by Mr. Nixon, and put forward alternative solutions of the problem.

The course adopted by Signor Gini is, in effect, to compare the increase in mortality with the probable error, so that if in two sections A and B of the populations, the rates of mortality are increased from m_1 and m_3 respectively, to m_2 and m_4, then the increase in A must be regarded as > or < the increase in B according as

$$\frac{m_2 - m_1}{\sqrt{m_1(1-m_1)}} \text{ is } > \text{ or } < \frac{m_4 - m_3}{\sqrt{m_3(1-m_3)}}$$

The objections to the criterion proposed by Gini are, I think, obvious. They are put with great force by Prof. Cantelli on pages 13–15 of his pamphlet, to which I would refer anyone who is interested in the matter.

Prof. Cantelli points out that the criterion proposed by Nixon cannot give a satisfactory answer to the question: "Which classes " of a population are the most seriously affected by the indirect " effects of a War"? For "Consider a case—fictitious or not does " not matter—which exposes the weakness of this criterion. Two " populations, A and B, have suffered severely from the effects of " a War. In the population A the mortality has increased from " 10 per thousand to 100 per thousand, while in the population B " it has increased from 10 per hundred to 100 per 100. Which of " the two populations has been more severely hit by the War? " That is, which population has suffered the heavier mortality? " Clearly, it seems to me, population B, which has been com- " pletely wiped out by the effects of the War. But the expression " used by Nixon gives the same result for the two populations, " namely:

"
$$\frac{100-10}{10}$$

" But even excluding the above example, it does not seem to me " that the expression

$$\frac{q_{ng}-q_n}{q_n} \text{ or } \frac{m_{ng}-m_n}{m_n}$$

" can answer questions of the kind asked above, since it, as we see " from its very nature, ignores the question of the exposed to risk."

Prof. Cantelli goes on to show that this question can only be answered by a comparison, age-group by age-group, of the values of q_g, where q_g is the probability of death on the assumption that the exposed to risk are only subject to mortality caused by the indirect effects of the war. He finds the value of q_g as follows:

" Let us suppose that out of l persons exposed to risk m die in " one year from all causes, including the indirect effects of war. " Let us further suppose that we know that if no deaths were due " to the effects of war and that only the ordinary rate of mortality " was in operation n persons would die in the year. We can then " say that during the whole year of risk, $l-n$ persons are not liable " to die from normal causes. Hence it follows that the $m-n$ " persons, who die out of the $l-n$ under consideration, die from " causes due solely to the war, *i.e.*,

"
$$q_g = \frac{m-n}{l-n}$$

" But
$$\frac{m}{l} = q_{ng}; \quad \frac{n}{l} = q_n$$

"
$$\therefore \quad q_g = \frac{q_{ng}-q_n}{1-q_n} \qquad "$$

In a foot-note the author points out that this formula can be derived directly from Karup's theorem

$$p_{ng} = p_n \times p_g .$$

In order to compare the results given by his formula with those given by the formulæ of Hersch and Nixon, Gini calculated values of q_n from the mean annual number of deaths in the decennium 1876–85 and the census of 1880, and gives a series of tables, for the age-groups used by Prof. Hersch, for Switzerland, Holland and Belgium, distinguishing in each case between males and females.

I reproduce here Gini's tables for Swiss males, and have added, for purposes of comparison, a table of q_g calculated according to the formula given by Prof. Cantelli.

The indirect effect of the Franco-Prussian War, 1870–71, *on the Mortality of the Swiss Male Population.*

Age Group	The absolute excess of Mortality, namely, $10{,}000\,(q_{ng} - q_n)$ Prof. Hersch's method (1)	Normal Mortality, namely, $10{,}000\,q_n$ as calculated by Signor Gini (see above) (2)	Percentage increase in Mortality $\frac{q_{ng} - q_n}{q_n}$ Mr. Nixon's method (3)	Relative increase in Mortality $\frac{m_2 - m_1}{\sqrt{m_1(1-m_1)}}$ Gini's method (4)	$q_g = \frac{q_{ng} - q_n}{1 - q_n}$ Prof. Cantelli's method (5)
0–1	424	2,583	16	·097	·05716
1–5	74	205	36	·052	·00755
5–10	36	61	59	·046	·00362
10–15	11	35	31	·018	·00110
15–20	13	49	27	·018	·00130
20–25	65	73	89	·077	·00653
25–30	69	87	79	·074	·00696
30–35	38	99	38	·038	·00384
35–45	40	125	32	·036	·00405
45–55	51	198	26	·037	·00520
55–65	70	369	19	·037	·00726
65–75	132	774	17	·049	·01430
75–	252	1,722	15	·067	·03044

It will be noticed that this latter table supports the conclusion drawn by Prof. Hersch, and militates against those drawn by Mr. Nixon.

Yours faithfully,

D. S. SAVORY.

3, *Queen's Gardens,*
Ealing, W. 5.

INDUSTRIAL MORTALITY IN 1915-1917.

To the Editors of the Journal of the Institute of Actuaries.

DEAR SIRS,—Recently I had occasion to investigate the mortality experience of a very large number of policies on male lives embodying industrial assurance contracts of various descriptions.

The general results obtained were sufficiently remarkable, I think, to be of interest to readers of the *Journal.* I have therefore, in the accompanying diagram, exhibited them graphically in comparison with two standard population curves, namely, Dr. Farr's English Life Table No. 3 and the English Life Table No. 8 published in the supplement to the 75th Annual Report of the Registrar-General. The function chosen for comparison is the rate of mortality q_x as shown by the following experience of males :

Rate of mortality experienced among industrial policies during 1913
,, ,, ,, ,, ,, ,, 1915
,, ,, ,, ,, ,, ,, 1916
,, ,, ,, ,, ,, ,, 1917
,, as shown by the English Life Table No. 8.
,, ,, ,, ,, ,, No. 3.

The exposed to risk and deaths were obtained from the valuation class books which were very convenient for the purpose, the policies being tabulated on the assumption that on the average exact ages were attained at the date of valuation.

This method gives a possible range of nearly two years in the tabulated age, *i.e.*, the assumed exact age x represents ages between $x-1$ and $x+1$. Apart from abnormal disturbances in the law of mortality this method gives results reliable enough for most purposes. When, however, a section of the experience is subjected to a very violent change in the progression of the rates of mortality over a series of ages a certain amount of error is introduced. In the present instance the method of classification undoubtedly has the effect of over-estimating the rates of mortality at ages from 16 to 20, and somewhat under-estimating them thereafter.

The 1913 curve shows a close correspondence with that of the general population particularly at the military ages.

The curves for the three complete war years 1915, 1916, and 1917 each exhibit certain special features, and it is believed are the only published statistics showing the incidence of the War on the mortality of a section of the community sufficiently large to be representative of the whole male population.

The experiences of separate years were large enough to give results that did not require graduation to enable a reliable comparison to be made between them, and no such adjustment has been attempted. The close correspondence of the 1913 curve with the English No. 8 curve is sufficient evidence in support of this statement.

The remarkable feature of the 1915 curve is the continuance of high rates at ages over 40. The 1916 curve, while showing the

same main features as that for 1915, shows lower rates than the 1913 curve at ages over 40. The causes that gave rise to these features suggest some interesting speculations as to ages of men in the "Old Army" and the effect of war conditions on the vitality of the general population. I have not the necessary statistics available however to investigate these points. The 1917 curve exhibits two features of interest, namely, the moving of the maximum point to an age one year younger, accompanied by a reduction in the rate of mortality below age 20, that is, at the training ages.

The curves of course do not show the relative rates of mortality among combatants in the three years under review, but I think it may be claimed that they are a fair index of the toll that the War has taken of the manhood of the nation during that period.

I am, Dear Sirs,

Yours faithfully,

J. BURN.

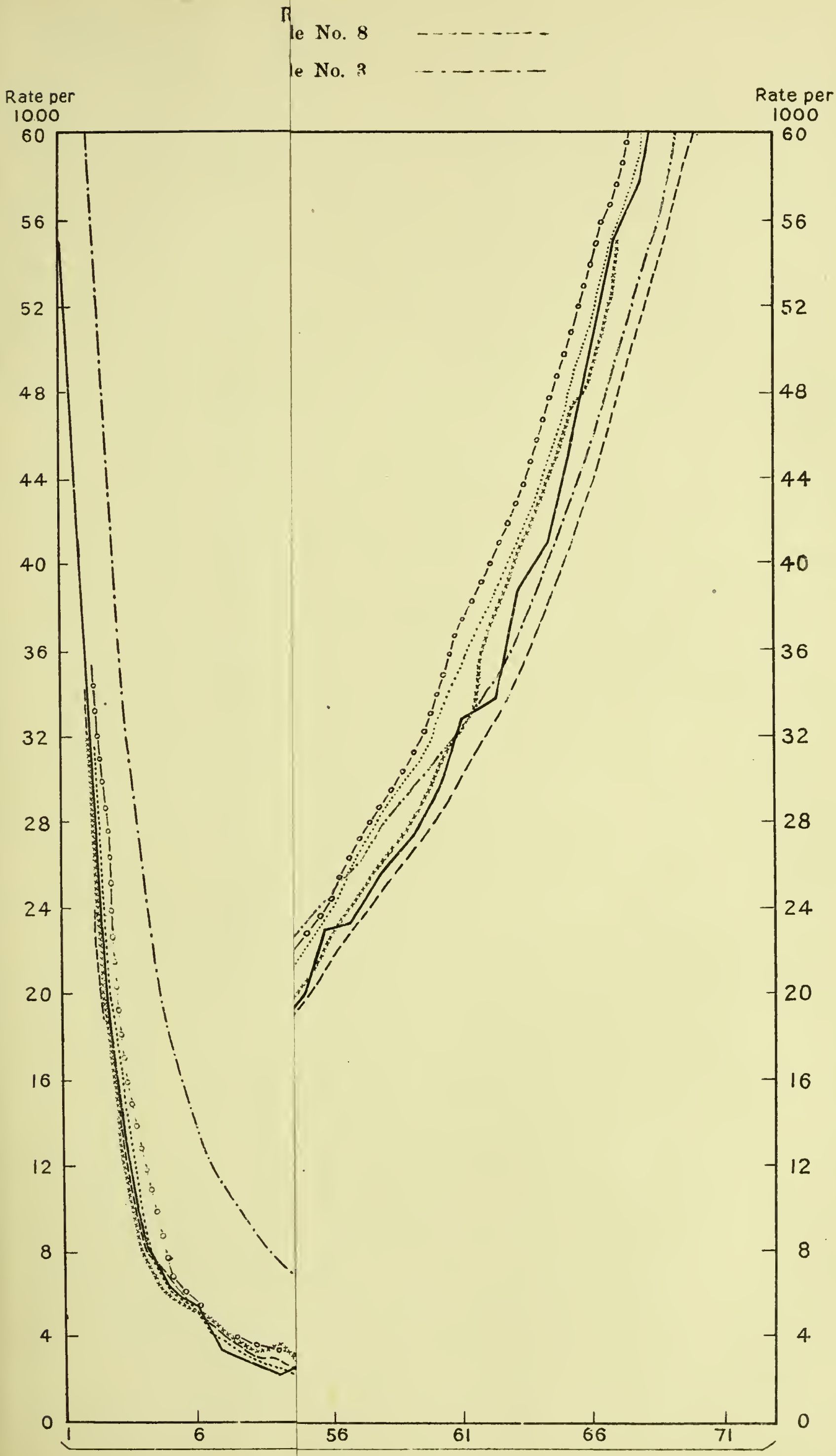
Prudential Assurance Co., Limited,
Holborn Bars, E.C. 1.
20 *March* 1918.

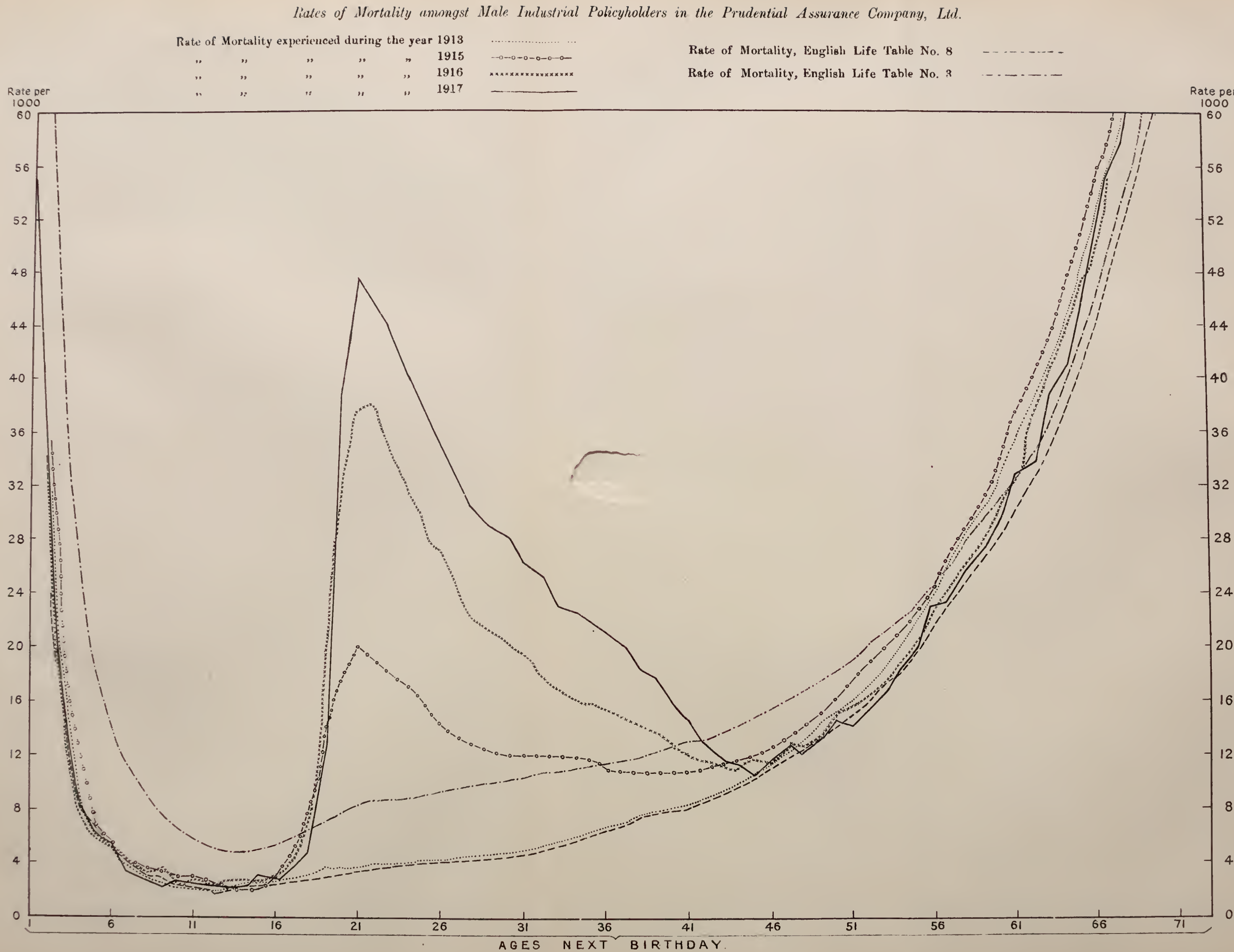
*** Mr. Burn has kindly supplied the actual rates of mortality experienced by the Prudential at ages 16 to 60 in the years 1913 and 1915–1917. These rates are given in the following Table, with the addition (in the first column) of the English Life No. 8 rates :

Values of q_x. *Ages* 16–60. *Male Lives.*

Age	English Life No. 8	Prudential Experience			
		1913	1915	1916	1917
16	·00259	·00290	·00467	·00404	·00374
17	·00279	·00317	·00790	·00840	·00483
18	·00302	·00379	·01386	·01856	·01283
19	·00326	·00372	·01845	·03201	·03834
20	·00348	·00397	·02008	·03731	·04742
21	·00366	·00386	·01838	·03772	·04559
22	·00378	·00386	·01695	·03460	·04394
23	·00386	·00377	·01665	·03124	·04063
24	·00392	·00399	·01534	·02805	·03769
25	·00400	·00441	·01376	·02572	·03507
26	·00411	·00442	·01340	·02379	·03248
27	·00425	·00451	·01311	·02192	·03014
28	·00440	·00475	·01199	·02098	·02886
29	·00458	·00532	·01246	·02001	·02794
30	·00478	·00499	·01168	·01915	·02601

any, Ltd.


R
le No. 8
le No. 3
Rate per 1000
Rate per 1000
60
56
52
48
44
40
36
32
28
24
20
16
12
8
4
0
1
6
56
61
66
71



Rates of Mortality amongst Male Industrial Policyholders in the Prudential Assurance Company, Ltd.
Rate of Mortality experienced during the year 1913
" " " " " 1915
" " " " " 1916
" " " " " 1917
Rate of Mortality, English Life Table No. 8
Rate of Mortality, English Life Table No. 3
Rate per 1000
0
4
8
12
16
20
24
28
32
36
40
44
48
52
56
60
1
6
11
16
21
26
31
36
41
46
51
56
61
66
71
AGES NEXT BIRTHDAY

Values of q_x. Ages 16–60. *Male Lives—continued.*

Age	English Life No. 8	Prudential Experience 1913	1915	1916	1917
31	·00502	·00562	·01182	·01773	·02501
32	·00528	·00520	·01177	·01675	·02304
33	·00558	·00605	·01225	·01576	·02265
34	·00590	·00594	·01145	·01565	·02194
35	·00624	·00642	·01118	·01488	·02097
36	·00659	·00667	·01087	·01440	·01977
37	·00695	·00714	·01069	·01369	·01816
38	·00731	·00742	·01065	·01286	·01744
39	·00769	·00811	·01130	·01216	·01588
40	·00811	·00798	·01119	·01152	·01452
41	·00858	·00902	·01094	·01140	·01286
42	·00909	·00908	·01137	·01099	·01159
43	·00964	·01005	·01123	·01065	·01111
44	·01024	·01063	·01169	·01173	·01043
45	·01089	·01121	·01258	·01130	·01154
46	·01158	·01184	·01288	·01253	·01236
47	·01231	·01308	·01438	·01233	·01205
48	·01308	·01391	·01469	·01316	·01290
49	·01391	·01520	·01660	·01479	·01427
50	·01482	·01609	·01781	·01551	·01413
51	·01586	·01717	·01831	·01643	·01533
52	·01701	·01877	·01873	·01710	·01637
53	·01827	·01861	·02107	·01898	·01854
54	·01963	·02269	·02255	·02008	·01987
55	·02111	·02423	·02372	·02234	·02283
56	·02272	·02637	·02642	·02397	·02318
57	·02444	·02852	·02869	·02629	·02558
58	·02629	·02991	·03191	·02786	·02731
59	·02827	·03307	·03559	·03075	·02996
60	·03042	·03462	·03802	·03268	·03291

THE INSTITUTE OF ACTUARIES.

The Council of the Institute have recently had under consideration the position of candidates for the Examinations in view of the situation created by the War. In order to afford candidates every possible help, under the special conditions which have arisen, the Council have decided:

(1) To reduce the number of subjects of examination, and in this way to restrict the scope of the necessary reading,

(2) To request the Board of Examiners to afford candidates some guidance in the reading required; and

(3) To hold Examinations more frequently.

They have, therefore, suspended the existing Regulations and Syllabus of Examinations, and have adopted, until further notice, the appended revised Regulations and Syllabus, Recommendations by the Board of Examiners as to the reading required for the Examinations are in course of preparation,

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REGULATIONS and SYLLABUS of Examinations for admission to the Classes of Student, Associate, and Fellow.

(*These Regulations will continue in force for not less than four years from the date of the first Examinations held after the termination of the War. All previous Regulations are hereby suspended.*)

1. The Examinations held by the Institute are as set out in the annexed Syllabus.

2. An applicant for admission to the Class of Student must (*a*) have furnished such evidence of general education as the Council may from time to time prescribe, and (*b*) have passed Part I of the Examinations under these or previous Regulations.

As evidence of general education the Council will (subject as hereinafter provided) require a Certificate showing that the applicant has passed the Matriculation or Senior School Examination of the University of London, or a similar Examination of any University in the British Empire, or the Oxford or Cambridge Senior Local Examination, or some other Examination approved by the Council from time to time or accepted on individual application. Provided that if in any case the Council are satisfied that an applicant has been precluded by special circumstances from passing one of the Examinations hereinbefore mentioned they may, if otherwise satisfied as to his general education, dispense with a Certificate of his having passed such Examination.

3. An applicant for admission to the Class of Associate must (*a*) be a Student and (*b*) have passed Parts I and II of the Examinations under these or previous Regulations.

4. An applicant for admission to the Class of Fellow must (*a*) be a Student or Associate and (*b*) have passed Parts I, II, III, and (subject as hereinafter provided) Part IV of the Examinations under these or previous Regulations. Provided that an applicant who has passed Parts I and II under previous Regulations shall be required to pass only Part III and Part IV, Section A, under these Regulations, and that an applicant who has passed Parts I, II, and III

under previous Regulations shall be required to pass only Part IV, Section A, under these Regulations.

5. An applicant who, under the Bye-laws or previous Regulations, has been exempted from passing a specified Part of the Examinations, shall be considered, for the purposes of these Regulations, to have passed that Part.

6. An applicant for admission to any Class shall, after having passed the prescribed Examinations, be admitted to that Class on signing the proper Form of Obligation or of Transfer, as the case may be, and on paying the subscription of the Class for the current year.

7. A Candidate shall not (except as hereinafter provided) present himself for Examination in a Part or a Section thereof until he has passed the previous Part. Provided that a Candidate, who is a Graduate in Mathematical Honours of a University in the British Empire, or any other Candidate on the recommendation of a Tutor of the Institute or of two Fellows having personal knowledge of his qualifications, may take Part I (or the remaining Section of Part I if he shall have passed in one Section) and Part II at the same Examination, but no such Candidate shall be considered to have passed Part II until he shall have passed both Part I and Part II.

8. The Fee for a Part of the Examination is £1. 1*s*., and for one Section of a Part 10*s*. 6*d*.

9. A Candidate will not be allowed to present himself for Examination until he has paid all entrance fees, subscriptions, and examination fees that may be due and complied with the requirements of the Bye-laws and these Regulations.

10. Examinations will be held in London and at such other places as the Council may appoint, in June and December of each year, or at such other times as the Council may prescribe. The first Examinations under these Regulations will be held as soon as possible after the termination of the War.

At least four months' notice will be given by public advertisement of the places at which Examinations will be held and of the dates of such Examinations.

A Candidate for Examination at any place in the United Kingdom must give notice in writing to the Assistant Secretary so that such notice shall reach the Assistant Secretary at least fourteen days before the date of Examination, and a Candidate for Examination at any place outside the United Kingdom must give such notice so that it shall reach the Assistant Secretary at least two months before the date of Examination. At the time of giving notice a Candidate must specify the Part or Section of a Part for which he intends to present himself, and must pay the prescribed fee.

11. The names of successful Candidates in each Part or Section of a Part will be arranged in alphabetical order, without distinction of Class.

March 1918.

SYLLABUS OF EXAMINATIONS.

Part I.

Section A.

Arithmetic and Algebra; the theory and use of Logarithms; the Elements of the Theory of Probabilities.

The Elements of the Calculus of Finite Differences, including Interpolation and Summation; Elementary Differential and Integral Calculus, excluding questions necessitating the use of Trigonometry.

Section B.

Compound Interest and Annuities-Certain, including the construction and use of relative Tables.

A Candidate must pass in both Sections. The two Sections may be taken at the same Examination, or either Section may be taken at one Examination and the remaining Section at a subsequent Examination.

Two Papers will be set in Section A and One in Section B. Three hours will be allowed for each Paper.

Part II.

Life Contingencies, including Life Annuities and Assurances, and the construction and use of the Life-Table and monetary and other Tables based thereon; excluding questions on the compilation of Tables from Statistics, or on Graduation.

Either two or three Papers will be set at the discretion of the Examiners. Three hours will be allowed for each Paper.

Part III.

Section A.

The Methods of compiling Mortality, Sickness, Accident, and other similar Statistics, of deducing Tables therefrom, and of adjusting or graduating such Tables.

The Distinctive Features of existing Tables now in general use.

Section B.

The Valuation of the Liabilities of Life Assurance and Employers' Liability Insurance Companies.

The Principles and Methods of the Distribution of Surplus.

A Candidate must pass in both Sections. The two Sections may be taken at the same Examination, or either Section may be taken at one Examination and the remaining Section at a subsequent Examination.

Two papers will be set in each Section. Three hours will be allowed for each Paper.

Part IV.

Section A.

The general provisions of the Acts relating to Life Assurance Companies, Employers' Liability Insurance Companies and Friendly Societies.

The practical valuation of Life Interests and Reversions, and of Policies for Surrender or Purchase.

Section B.

The law relating to Life Assurance Contracts.

Life Assurance Accounts ; preparation of Schedules, Statements and Reports ; drafting of Policies and endorsements ; and other practical matters arising in Life Office Administration.

The determination of Office Rates of Premium for Assurances and Annuities, but excluding rates of Contribution for Sickness, Pension and Widows' and Orphans' Funds.

Extra premiums for under-average lives, hazardous occupations and residence in unhealthy climates ; and the materials available for their determination.

Section C.

The formation and valuation of and calculation of rates of contribution for Friendly Societies, Pension Funds, and Widows' and Orphans' Funds.

The National Health Insurance Acts, with special reference to their financial provisions.

The Elements of Statistics ; Official Statistics, with special reference to their employment in the solution of Actuarial Problems.

A Candidate must pass in Section A and (subject to the provision in Rule 4) in either Section B or Section C. Section A and Section B (or Section C) may be taken at the same Examination, or either Section may be taken at one Examination and the remaining Section at a subsequent Examination.

Two Papers will be set in each Section. Three hours will be allowed for each Paper.

Regulations for Probationers.

Probationers, while not being Members of the Institute, are allowed the following privileges, namely :

They are entitled to join the classes for Students, in accordance with the rules prescribed for such classes, and to attend the Ordinary General Meetings of the Institute, but not to vote or take part in the discussions thereat.

They may borrow books from the Library for the purposes of their studies, but this privilege is subject to the discretion of the Librarians, and to the rules which the Council may from time to time prescribe.

Application to become a Probationer must be made on a prescribed form, and the applicant must satisfy the Council as to his general education by furnishing evidence thereof similar to that required from an applicant for admission to the Class of Student. If the application is approved the applicant shall become a Probationer on payment of an entrance fee of 10*s*. 6*d*., but the Council may at any time withdraw their approval, and thereupon he shall cease to be a Probationer. Should a Probationer subsequently be admitted a Member of the Institute, the fee of 10*s*. 6*d*. paid by him on becoming a Probationer, will be taken as paid on account of the entrance fee as Student.

The annual subscription for Probationers is 10*s*. 6*d*., payable on admission and on 1st October in each year. If the subscription for any year be not paid before the 31st December, then the defaulter shall no longer be a Probationer.

Obituary.

HARRY ORRELL, Student of the Institute, Private, 20th Battalion, Manchester Regiment.

Killed in Action 23 *April* 1917.

BRIAN NEEDELL, Student of the Institute, Sergeant, 5th Battalion, City of London Regiment.

Killed in Action 2 *May* 1917.

WALTER ERNEST SMITH, Probationer of the Institute, Captain, King's Own Yorkshire Light Infantry.

Died of Wounds 5 *July* 1917.

SIDNEY GEORGE WEATHERDON, Probationer of the Institute, 2nd Lieut., 11th Battalion, Essex Regiment.

Killed in Action 19 *September* 1917.

THOMAS NOWELL ASKWITH, Student of the Institute, 2nd Lieut., Royal Field Artillery.

Killed in Action 26 *October* 1917.

GERALD DANBY DOUCET, Fellow of the Institute, 2nd Lieut., 7th Battalion, Northumberland Fusiliers.

Killed in Action 26 *October* 1917.

HUGH JERROLD HAMMOND, Student of the Institute, Captain, 12th Battalion, Royal Warwicks (attached 2nd Battalion, Gloucestershire Regiment).

Died of Wounds 23 *March* 1918.

WILLIAM ASKHAM, Probationer of the Institute, Lieut. (Acting Captain), Army Cyclist Corps.

Killed in Action 11 *April* 1918.

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Newton's Interpolation Formulas. *By* Duncan C. Fraser, M.A., F.I.A.

Following upon a suggestion which was made in a previous number of the *Journal* (vol. l, p. 20), an endeavour has been made in the following pages to bring together the whole of Newton's work on the subject of interpolation by means of formulas of finite differences. His contributions to this subject are comprised in three items (1) the "Methodus Differentialis"; (2) a letter written in 1675, giving detailed instructions for the calculation of certain tables; and (3) the celebrated Lemma No. 5 in Book III of the "Principia."

There are also various references to the subject in the "Commercium Epistolicum", a collection of letters relating to the controversy between Newton and Leibnitz as to the origin of the differential calculus.

Methodus Differentialis.—This is a short treatise, complete in itself, on central formulas of interpolation and their applications. It was first published in the year 1711 by William Jones in a volume in which he collected a number of Newton's shorter works under the title "Analysis Per "Quantitatum Series, Fluxiones ac Differentias: cum "Enumeratio Linearum Tertii Ordinis."

A photographic reproduction of the original Latin text,*

* The scale of the original has been reduced by one-fifth to admit of inclusion in the *Journal* page.

taken from a copy of the first edition in the Institute Library, and a translation are given. It is believed that no previous translation of this little work has been published.

Although the "Methodus Differentialis" was not printed until the year 1711, it was composed many years earlier. In the Latin preface (from which I translate), William Jones says:

> "The book is brought to a graceful close by the "addition of a little tract entitled 'Methodus Differen-"tialis', which I have transcribed by permission of the "distinguished author from his own autograph. . . . This "'Methodus Differentialis' depends upon the problem of "drawing a Parabolic Curve through a given number of "points, reference to which had been made by the "distinguished author in his letter to Oldenburg, sent in "1676, and a solution of which he gave in Lemma 5, "Book III of his 'Principia' by means of a construction "which is not at all the same as that which we now "present."

The letter to which William Jones refers is a letter dated 24 October 1676, which is included in the "Commercium Epistolicum." In the course of the letter Newton describes a method by which the function $\sqrt{a^2-ax+\frac{x^2}{2}}$ might be expanded in a series of powers of x; and then goes on to say (I translate freely from the original Latin), "But I attach "little importance to this method because when simple series "are not obtainable with sufficient ease, I have another "method not yet published by which the problem is easily "dealt with. It is based upon a convenient, ready and general "solution of this problem, *To describe a geometrical curve which "shall pass through any given points.*"

He then refers to cases in which such a problem can be solved by geometrical constructions without calculation; and adds: "but the above problem is of another kind; and "although it may seem to be intractable at first sight, it is "nevertheless quite the contrary; perhaps indeed it is one of "the prettiest problems that I can ever hope to solve." This fixes the date of composition of the "Methodus" as prior to October 1676; and there is some reason to think that its date may be several years earlier.

A short account of the "Methodus Differentialis" has previously been given in the *Journal* (vol. xv, pp. 145 and 177),

by Professor Ludwig Oppermann, of Copenhagen. He there gives the date of publication in error as 1715. He gives his opinion that this little treatise was written many years before the Lemma, basing his view apparently on internal evidence only. The preface, which is conclusive as to the priority of the "Methodus", is not printed in Horsley's complete edition of Newton's works and may not have been seen by Professor Oppermann.

NOTES.

In *Proposition I* it is shown that if the ordinate corresponding to the abscissa $A+x$ is $a+bx+cx^2+dx^3+ex^4+\ldots$ then expressions for all the divided differences can be exactly obtained. This is proved by actual division for the case when the highest power of x involved in the expression for the ordinate is the fourth.

Proposition II.—In the same case as in Proposition I, the values of five ordinates being known, full directions are given for the solution of the five simultaneous equations, from which can be obtained the values of the coefficients in terms of an ordinate and of divided differences of the ordinates.

In these two Propositions there is some confusion in the original text as to the first term in the expression for the ordinate. In the enunciation of Proposition I it is omitted, and I have supplied it. In the Table it is given as A, the abscissa being $A+x$. In the demonstrations it is not mentioned until the end of Proposition II, and an error occurs there, it being stated that the final operation in the solution of the simultaneous equation gives the first term of the abscissa A. It is quite clear that what is obtained is the first term of the ordinate, and in the translation this has been called a in correspondence with the remaining coefficients b, c, d, &c.

Proposition III. Case I.—The ordinates being equidistant and the number being odd a central difference formula is given in terms of the central ordinate and the central differences which are in line with it. The coefficients of the formula are

$$1,\ x,\ \frac{1}{2}x^2,\ \frac{x(x^2-1)}{6},\ \frac{x(x^2-1)}{24},\ \&c.,$$

and it will be recognized that this is the formula which is

commonly called Stirling's. The differences used in both cases of this proposition are not divided differences but simple differences without division. It will be found on examination that Newton takes his differences and measures the values of x in a sense opposite to that which is now customary. To bring the details of the work into conformity with our present practice the signs of the odd powers of x and the signs of the odd differences would have to be altered; but as it happens that the odd powers of x always occur in combination with odd differences, this makes no difference in the formula.

Case II.—The ordinates being equidistant and their number being even, a formula is given in terms of the mean of the two central ordinates and of the central differences opposite to that mean. The coefficients of the formula are

$$1,\ x,\ \frac{4x^2-1}{8},\ \frac{x(4x^2-1)}{24},\ \frac{(4x^2-1)(4x^2-9)}{384},\ \&c.$$

This formula is now commonly known by the name of Bessel's formula. There is a misprint in the original, e_2+e_3 being printed for $\frac{e_2+e_3}{2}$.

Proposition IV.—A misprint occurs in the original, where the expression for the difference $b6$ ought to be $\frac{\text{A6B6}+\text{A7B7}}{\text{A6A7}}$ and not $\frac{\text{A6B6}-\text{A7B7}}{\text{A6A7}}$. Newton uses the expressions A6, B6, &c., to represent the arithmetical values of the lengths of the ordinates without reference to sign, and an example of the same practice will be found in the Lemma.

Case I.—An odd number of ordinates being given at points on the abscissa $\alpha, \beta, \gamma, \delta$, &c., which are separated by unequal intervals, a central difference formula is given in terms of the ordinate at the central point δ and of the central divided differences which are in line with it, the coefficients being

$$1,\ x-\delta,\ (x-\delta)\times\frac{1}{2}\left\{\begin{matrix}(x-\gamma)\\+(x-\epsilon)\end{matrix}\right\},\ (x-\delta)(x-\gamma)(x-\epsilon),$$

$$(x-\delta)(x-\gamma)(x-\epsilon)\times\frac{1}{2}\left\{\begin{matrix}(x-\beta)\\+(x-\zeta)\end{matrix}\right\},\ \&c.$$

Case II.—The number of ordinates being even and the two central ordinates being at the points δ and ϵ, a formula is given in terms of the mean of the two central ordinates and of the divided differences in line with that mean. The coefficients are

$$1,\ \frac{1}{2}\left\{\begin{matrix}(x-\delta)\\+(x-\epsilon)\end{matrix}\right\},\ (x-\delta)(x-\epsilon),\ (x-\delta)(x-\epsilon)\times\frac{1}{2}\left\{\begin{matrix}(x-\gamma)\\+(x-\zeta)\end{matrix}\right\},$$

$$(x-\delta)(x-\epsilon)(x-\gamma)(x-\zeta),\ \&c.$$

The analogy of these two formulas for divided differences with Stirling's formula and Bessel's formula will be easily seen.

In *Proposition V* Newton points out the application of the above four formulas when it is required to find any intermediate term of a series, of which certain terms are given.

In *Proposition VI* he points out that approximate expressions for the area of a curve, of which certain ordinates are known, can be derived from the preceding formulas.

In the *Scholium* Newton gives well-known formulas for the bisection of an interval, and for finding the area, when four ordinates are known. He then goes on to describe a process by which the problem of finding the approximate area when $2n+1$ ordinates are known can be reduced to the case of finding the area in terms of $n+1$ ordinates. It will be found on examination that Newton's process amounts to exactly the same thing as applying the formula for $n+1$ ordinates separately to the two halves of the curve of which $2n+1$ ordinates are given.

The meaning of his next paragraph is not entirely clear, but Newton's idea may have been to simplify the process of finding the approximate area by taking the sums of the ordinates in two's or three's, &c., using these sums as new ordinates and passing through their extremities a new curve, the area of which, taken between suitable limits, would approximate to the area required.

Letter on the Construction of Tables.—The two letters here printed, the first of which is simply a letter from William Jones to Professor Cotes, enclosing a letter from Newton, dated 8 May 1675, to Mr. John Smith, are taken from a

volume published by J. Edleston, M.A., in 1850, and entitled "Correspondence of Sir Isaac Newton and Professor Cotes, "including letters of other eminent men, now first published "from the originals in the Library of Trinity College, "Cambridge", &c. The directions for the construction of tables given by Newton are of a very practical character, and will be easily followed by anyone who wishes to examine his method in detail.

The formulas employed can readily be obtained by applying the binomial theorem and by using Stirling's interpolation formula.

The two principal formulas, namely:

$$s=\omega+\frac{1}{2}st+\frac{1}{6}m$$

$$\zeta=\frac{\omega}{10}+\frac{\frac{1}{2}st}{100}+\frac{m}{6000}$$

expressed in modern notation are as follows:

$$F(x+1)-F(x)=F'(x)+\frac{1}{2}F''(x)+\frac{1}{6}F'''(x)$$

$$F\left(x+\frac{1}{10}\right)-F(x)=\frac{1}{10}F'(x)+\frac{1}{2}\cdot\frac{1}{100}F''(x)+\frac{1}{6}\cdot\frac{1}{1000}F'''(x)$$

The last term in this second formula, added by Mr. Edleston for the sake of completeness, appears to be superfluous.

It will be noticed that they are in fact formulas of the differential calculus and can be written down at once from Taylor's theorem. But Newton's method of obtaining them was more probably that suggested above. Brook Taylor, the discoverer of Taylor's theorem, was not born until 1685, and the theorem, which he obtained as a simple collorary to Newton's descending difference formula by making the differences indefinitely small, was first published in 1715.

Lemma No. V, Book III of the "Principia."—This has been previously translated in an English version of the "Principia" published by Motte in 1729. The version here given is new.

The Lemma gives the well-known propositions of

interpolation by means of descending differences, for equal and for unequal intervals, which have always been regarded as laying the foundation of the calculus of finite differences. It appears from Newton's own statements that the whole of the "Principia" was written between December 1684 and May 1686, with the exception of 14 specified propositions among which the Lemma is not included. The date of the composition of the Lemma was therefore shortly before May 1686. It is difficult to suppose that he was not previously aware of the propositions stated in the Lemma; and it is remarkable that he had not included them in the "Methodus" which was composed many years before. The explanation may be that at the time he wrote the "Methodus" his mind was much engrossed with schemes for the calculation of extensive tables, for which the formulas of central differences were of greater practical use than formulas proceeding by descending differences. At the time he composed the Lemma, the particular point he had in view, as will be found by reference to the immediately succeeding proposition in the "Principia", was its application to an isolated case of interpolation.

In Newton's letter to Mr. John Smith, the notation has been reproduced without change, and it may be necessary to warn the reader that in such symbols as 2F and F2 the 2 is a suffix merely. In the versions given of the "Methodus" and of the Lemma, suffixes have been printed in the way now usual.

I have to express my acknowledgments to Mr. Walter Stott and Mr. R. O'Donovan, both of the Royal Insurance Company, Ltd., for their valuable assistance in the translation of the "Methodus." For the final form of that translation and for any defects which may be found in it, and for the translation of the Lemma, &c., I must take the entire responsibility.

In a future number of the *Journal* I hope to discuss the references to the subject of these notes which are to be found in the letters of Newton included in the "Commercium Epistolicum."

Reference may appropriately be made here to the valuable and interesting historical notes included in a

contribution to vol. xviii of the Transactions of the Actuarial Society of America, by Mr. S. A. Joffe, under the title "Interpolation Formulæ and Central-Difference Notation", in which he traces the history of the subject from the time of Newton, and draws particular attention to the connection between Newton's general formula for unequal differences and the general interpolation formulæ of Euler and Lagrange.

METHODUS DIFFERENTIALIS.

PROP. I.

S*I figuræ curvilineæ Abſciſſa componatur ex quantitate quavis data* A, *&* *quantitate indeterminata* x, *& Ordinata conſtet ex datis quotcunque quantitatibus* b, c, d, e, *&c. in totidem terminos hujus progreſſionis Geometricæ* x, x^2, x^3, x^4, *&c. reſpective ductis, & ad Abſciſſæ puncta totidem data erigantur Ordinatim applicatæ : dico quod Ordinatarum differentiæ primæ dividi poſſint per earum intervalla, & differentiarum ſic divi-*

ſarum differentiæ dividi poſſint per Ordinatarum binarum intervalla, & harum differentiarum ſic diviſarum differentiæ dividi poſſint per Ordinatarum ternarum intervalla, & ſic deinceps in infinitum.

Etenim ſi pro Abſciſſæ parte indeterminata x ponantur quantitates quævis datæ p, q, r, s, t, &c. ſucceſſive, & ad Abſciſſarum ſic datarum terminos erigantur Ordinatæ α, β, γ, δ, ε, &c Hæ Abſciſſæ & Ordinatæ & Ordinatarum differentiæ diviſæ per Abſciſſarum differentias (quæ utique ſunt Ordinatarum intervalla) & quotorum differentiæ diviſæ per Ordinatarum alternarum differentias, & ſic deinceps, exhibentur per Tabulam ſequentem.

Abſciſſæ	Ordinatæ
$A+p$	$A+bp+cp^2+dp^3+ep^4=\alpha$
$A+q$	$A+bq+cq^2+dq^3+eq^4=\beta$
$A+r$	$A+br+cr^2+dr^3+er^4=\gamma$
$A+s$	$A+bs+cs^2+ds^3+es^4=\delta$
$A+t$	$A+bt+ct^2+dt^3+et^4=\varepsilon$
Diviſor. Diff. Ord.	**Quoti per diviſionem prodeuntes.**
$p-q)\ \alpha-\beta$	$b+c\times\overline{p+q}+d\times\overline{pp+pq+qq}+e\times\overline{p^3+p^2q+pq^2+q^3}=\zeta$
$q-r)\ \beta-\gamma$	$b+c\times\overline{q+r}+d\times\overline{qq+qr+rr}+e\times\overline{q^3+q^2r+qr^2+r^3}=\eta$
$r-s)\ \gamma-\delta$	$b+c\times\overline{r+s}+d\times\overline{rr+rs+ss}+e\times\overline{r^3+r^2s+rs^2+s^3}=\theta$
$s-t)\ \delta-\varepsilon$	$b+c\times\overline{s+t}+d\times\overline{ss+st+tt}+e\times\overline{s^3+s^2t+st^2+t^3}=\kappa$
$p-r)\ \zeta-\eta$	$c+d\times\overline{p+q+r}+e\times\overline{pp+pq+qq+pr+qr+rr}=\lambda$
$q-s)\ \eta-\theta$	$c+d\times\overline{q+r+s}+e\times\overline{qq+qr+rr+qs+rs+ss}=\mu$
$r-t)\ \theta-\kappa$	$c+d\times\overline{r+s+t}+e\times\overline{rr+rs+ss+rt+st+tt}=\nu$
$p-s)\ \lambda-\mu$	$d+e\times\overline{p+q+r+s}=\xi$.
$q-t)\ \mu-\nu$	$d+e\times\overline{q+r+s+t}=\pi$.
$p-t)\ \xi-\pi$	$e=\sigma$.

PROP.

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PROP. II.

Iiſdem poſitis, & quod numerus terminorum b, c, d, e, *&c. ſit finitus, dico quod Quotorum ultimus æqualis erit ultimo terminorum* b, c, d, e, *&c. et quod per Quotos reliquos dabuntur termini reliqui* b, c, d, e, *&c. et his datis dabitur Linea Curva generis Parabolici quæ per Ordinatarum omnium terminos tranſibit.*

Etenim in Tabula ſuperiore Quotus ultimus σ æqualis erat termino ultimo e. Et hic terminus ductus in ſummam datam $\overline{p+q+r+s}$, & ablatus de Quoto ξ relinquit terminum penultimum d. Et quantitates jam datæ $d\times\overline{p+q+r}+e\times\overline{pp+pq+qq+pr+qr+rr}$, ſi auferantur de Quoto λ, relinquunt terminorum antepenultimum c. Et quantitates jam datæ $c\times\overline{p+q}+d\times\overline{pp+pq+qq}+e\times\overline{p^3+ppq+pqq+q^3}$, ſi auferantur de Quoto ζ, relinquunt terminum b. Et ſimili computo ſi plures eſſent termini, colligerentur omnes per Quotorum Ordines totidem. Deinde quantitates datæ $bp+cpp+dp^3+ep^4$, ſi ſubducantur de Ordinata prima α, relinquunt Abſciſſæ terminum primum A. Et quantitas $A+bx+cx^2+dx^3+ex^4+$ &c. eſt Ordinata Curvæ generis Parabolici quæ per Ordinatarum omnium datarum terminos tranſibit, exiſtente Abſciſſa $A+x$.

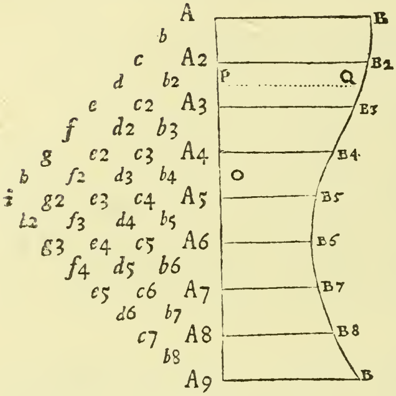
Ex his Propoſitionibus quæ ſequuntur facile colligi poſſunt.

PROP. III.

Si Recta aliqua AA9 *in æquales quotcunque partes* AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta diviſionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire curvam Geometricam generis Parabolici quæ per omnium erectarum terminos* B, B2, B3, *&c. tranſibit.*

Erectarum AB, A2B2, A3B3, &c. quære differentias Primas, b, $b2$, $b3$, &c. Secundas c, $c2$, $c3$, &c. Tertias d, $d2$, $d3$, &c. et ſic deinceps uſque dum veneris ad ultimam differentiam, quæ hic ſit i.

Tunc.



Tunc incipiendo ab ultima differentia excerpe medias differentias in alternis Columnis vel Ordinibus differentiarum, & Arithmetica media inter duas medias reliquarum, Ordine pergendo usque ad Seriem primorum terminorum AB, A2B2, A3B3, &c. sint hæc $k, l, m, n, o, p, q, r, s,$ &c. quorum ultimus significet ultimam differentiam; penultimus medium Arithmeticum inter duas penultimas differentias; antepenultimus mediam trium antepenultimarum differentiarum, & sic deinceps usque ad primum quod erit vel medius terminorum A, A2, A3, &c. vel Arithmeticus medius inter duos medios. Prius accidit ubi numerus terminorum A, A2, A3, &c. est impar; posterius ubi par.

C A S. I.

In Casu priori, sit A5B5 iste medius terminus, hoc est, A5B5 $= k$, $\frac{b4+b5}{2} = l$, $c4 = m$, $\frac{d3+d4}{2} = n$, $e3 = o$, $\frac{f2+f3}{2} = p$, $g2 = q$, $\frac{h+h2}{2} = r$, $i = s$. Et erecta Ordinatim applicata PQ, dic A5P $= x$; & duc terminos hujus Progressionis

$$1 \times \frac{x}{1} \times \frac{x}{2} \times \frac{x^2-1}{3x} \times \frac{x}{4} \times \frac{x^2-4}{5x} \times \frac{x}{6} \times \frac{x^2-9}{7x} \times \frac{x}{8} \times \frac{x^2-16}{9x} \times \frac{x}{10} \times \frac{x^2-25}{11x} \times \frac{x}{12} \times \frac{x^2-36}{13x} \text{ \&c.}$$

in se continuo; & orientur termini

$$1.\ x.\ \frac{x^2}{2}.\ \frac{x^3-x}{6}.\ \frac{x^4-x^2}{24}.\ \frac{x^5-5x^3+4x}{120}.\ \frac{x^6-5x^4+4x^2}{720}.\ \frac{x^7-14x^5+49x^3-36x}{5040}.\ \text{\&c.}$$

per quos si termini seriei $k, l, m, n, o, p,$ &c. respective multiplicentur, aggregatum factorum $k + xl + \frac{x^2}{2}m + \frac{x^3-x}{6}n + \frac{x^4-x^2}{24}o + \frac{x^5-5x^3+4x}{120}p +$ &c. erit longitudo Ordinatim applicatæ PQ.

C A S. II.

In Casu posteriori, sint A4B4, A5B5 duo medii termini, hoc est, sit $\frac{A4B4+A5B5}{2} = k$, $b4 = l$, $\frac{c3+c4}{2} = m$, $d3 = n$, $e2 + e3 = o$, $f2 = p$, $\frac{g+g2}{2} = q$, & $h = r$.

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& $b = r$. Et erecta Ordinatim applicata PQ, biseca $A4A5$ in O, & dicto OP $= x$, duc Terminos hujus Progressionis

$$1 \times \frac{x}{1} \times \frac{xx - \frac{1}{4}}{2x} \times \frac{x}{3} \times \frac{xx - \frac{9}{4}}{4x} \times \frac{x}{5} \times \frac{xx - \frac{25}{4}}{6x} \times \frac{x}{7} \times \frac{xx - \frac{49}{4}}{8x},$$ &c. in se continuo; et orientur termini $1.\ x.\ \frac{4xx-1}{8}.\ \frac{4x^3-x}{24}.\ \frac{16x^4-40x^2+9}{384}.$ &c. per quos si termini series k, l, m, n, o, p, q, &c. respective multiplicentur, aggregatum factorum $k + xl + \frac{4x^2-1}{8}m + \frac{4x^3-x}{24}n + \frac{16x^4-40x^2+9}{384}o +$ &c. erit Longitudo Ordinatim applicatæ PQ.

Sed hic notandum est quod intervalla AA2, A2A3, A3A4, &c. hic supponantur esse unitates, & quod differentiæ colligi debent auferendo inferiores quantitates de superioribus, A2B2 de AB, A3B3 de A2B2, $b2$ de b, &c. et faciendo ut sint $AB - A2B2 = b$, $A2B2 - A3B3 = b2$, $b - b2 = c$, &c. adeoque quando differentiæ illæ hoc modo prodeunt negativæ signa earum mutanda sunt.

PROP. IV.

Si recta aliqua in partes quotcunque inæquales AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta divisionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire Curvam Geometricam generis Parabolici quæ per omnium erectarum terminos* B, B2, B3, *&c. transibit.*

Sunto puncta data B, B2, B3, B4, B5, B6, B7, &c. et ad Abscissam quamvis AA7 demitte Ordinatas perpendiculariter BA, B2A2, &c.

Et fac $\frac{AB - A2B2}{AA2} = b$, $\frac{A2B2 - A3B3}{A2A3} = b2$,

$\frac{A3B3 - A4B4}{A3A4} = b3$, $\frac{A4B4 - A5B5}{A4A5} = b4$,

$\frac{A5B5 - A6B6}{A5A6} = b5$, $\frac{A6B6 - A7B7}{A6A7} = b6$,

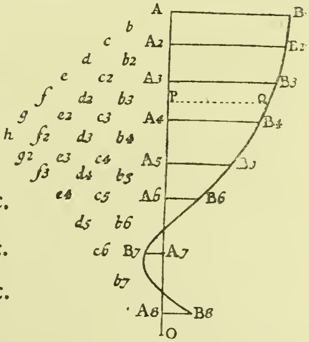
$\frac{-A7B7 - A8B8}{A7A8} = b7$.

Deinde $\frac{b - b2}{AA3} = c$, $\frac{b2 - b3}{A2A4} = c2$, $\frac{b3 - b4}{A3A5} = c3$, &c.

Tunc $\frac{c - c2}{AA4} = d$, $\frac{c2 - c3}{A2A5} = d2$, $\frac{c3 - c4}{A3A6} = d3$, &c.

Et $\frac{d - d2}{AA5} = e$, $\frac{d2 - d3}{A2A6} = e2$, $\frac{d3 - d4}{A3A7} = e3$, &c.

Sic pergendum est ad ultimam differentiam.



B b Differen-

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Differentiis sic collectis & divisis per intervalla Ordinatim applicatarum; in alternis earum Columnis sive Seriebus vel Ordinibus excerpe medias, incipiendo ab ultima, & in reliquis Columnis excerpe media Arithmetica inter duas medias, pergendo usque ad seriem primorum terminorum, AB, A2B2, &c. Sunto hæc k, l, m, n, o, p, q, r, &c. quorum ultimus terminus significet ultimam differentiam; penultimus medium Arithmeticum inter duas penultimas; antepenultimus mediam trium antepenultimarum, &c. Et primus k erit media Ordinatim applicata, si numerus datorum punctorum est impar; vel medium Arithmeticum inter duas medias, si numerus eorum est par.

C A S. I.

In Casu priori, sit A4B4 ista media Ordinatim applicata, hoc est, sit $A4B4 = k$, $\frac{b3+b4}{2} = l$, $c3 = m$, $\frac{d2+d3}{2} = r$, $e2 = o$, $\frac{f+f2}{2} = p$, $g = q$. Et erecta Ordinatim applicata PQ, & in Basi AA5 sumpto quovis puncto O, dic $OP = x$, & duc in se gradatim terminos hujus Progressionis

$$1 \times \overline{x - OA4} \times \overline{x - \frac{OA3 + OA5}{2}} \times \frac{\overline{x - OA3} \times \overline{x - OA5}}{x - \frac{1}{2}\overline{OA3 + OA5}} \times \overline{x - \frac{OA2 + OA6}{2}} \times \&c.$$

et ortam Progressionem asserva; vel quod perinde est duc terminos hujus Progressionis

$$1 \times \overline{x - OA4} \times \overline{\overline{x - OA3} \times \overline{x - OA5}} \times \overline{\overline{x - OA2} \times \overline{x - OA6}} \times \overline{\overline{x - OA} \times \overline{x - OA7}} \times \&c.$$

in se gradatim, & terminos exinde ortos duc respective in terminos hujus Progressionis

$$1.\ x - \frac{+OA3 + OA5}{2}.\ x - \frac{+OA2 + OA6}{2}.\ x - \frac{+OA + OA7}{2}, \ \&c.$$ et orientur termini intermedii tota Progressione existente

$$1.\ x - OA4.\ x^2 - \frac{+OA3 + 2OA4 + OA5}{2}x + \frac{OA3 + OA5}{2} \times OA4, \ \&c.$$

Vel dic $OA = \alpha$, $OA2 = \beta$, $OA3 = \gamma$, $OA4 = \delta$, $OA5 = \varepsilon$, $OA6 = \zeta$, $OA7 = \eta$: $\frac{OA3 + OA5}{2} = \theta$, $\frac{OA2 + OA6}{2} = \chi$, $\frac{OA + OA7}{2} = \lambda$. Et ex Progressione

$$1 \times \overline{x - \delta} \times \overline{\overline{x - \gamma} \times \overline{x - \varepsilon}} \times \overline{\overline{x - \beta} \times \overline{x - \zeta}} \times \overline{\overline{x - \alpha} \times \overline{x - \eta}}$$ &c. collige terminos quibus multiplicatis per $1.\ x - \theta,\ x - \chi,\ x - \lambda$, &c collige alios terminos intermedios, tota serie prodeunte

$$1,\ x - \delta,\ x^2 - \overline{\delta + \theta}x + \delta\theta,\ x^3 - \overline{\delta + 2\theta}x^2 + \overline{\gamma\varepsilon + 2\delta\theta}x - \gamma\delta\varepsilon, \ \&c.$$

per cujus terminos multiplica series k, l, m, n, o, &c. Et aggregatum productorum $k + \overline{x - \delta} \times l + \overline{x^2 - \overline{\delta + \theta}x + \delta\theta} \times m + \&c.$ erit longitudo Ordinatim applicatæ PQ.

C A S.

C A S. II.

In Caſu poſteriori, ſint $A4B4$, $A5B5$ duæ mediæ Ordinatim applicatæ, hoc eſt, $\frac{A4B4+A5B5}{2}=k$, $b4=l$, $\frac{c3+c4}{2}=m$, $d3=n$, $\frac{e2+e3}{2}=o$, $f2=p$, &c. Et alternorum k, m, o, q, &c. Coefficientes orientur ex multiplicatione terminorum hujus Progreſſionis in ſe

$$1\times\overline{\overline{x-OA4}\times\overline{x-OA5}}\times\overline{\overline{x-OA3}\times\overline{x-OA6}}\times\overline{\overline{x-OA2}\times\overline{x-OA7}}\times\overline{\overline{x-OA}\times\overline{x-OA8}}\ \&c.$$

Et reliquorum Coefficientes ex multiplicatione horum per terminos hujus Progreſſionis

$$x-\frac{+OA4+OA5}{2},\quad x-\frac{+OA3+OA6}{2},\quad x-\frac{+OA2+OA7}{2},\quad x-\frac{+OA+OA8}{2},\ \&c.$$

Hoc eſt, erit $k+\overline{x-\frac{+OA4+OA5}{2}}\times l+\overline{x^2-\overline{OA4+OA5}\,x+OA4\times OA5}\times m$, &c. Ordinatim applicata PQ,

vel $PQ=k+\overline{\begin{array}{l}x\\-\frac{1}{2}OA4\\-\frac{1}{2}OA5\end{array}}\times l+\overline{\begin{array}{l}x\\-OA4\end{array}}\times\overline{\begin{array}{l}x\\-OA5\end{array}}\times m+\overline{\begin{array}{l}x\\-OA4\end{array}}\times\overline{\begin{array}{l}x\\-OA5\end{array}}\times\overline{\begin{array}{l}x\\-\frac{1}{2}OA3\\-\frac{1}{2}OA6\end{array}}\times n$ &c.

$$\text{Sive dic } x-\frac{+OA4+OA5}{2}=\pi,\qquad \overline{x-OA4}\times\overline{x-OA5}=\rho,$$

$$\rho\times\overline{x-\frac{+OA3+OA6}{2}}=\sigma,\ \rho\times\overline{x-OA3}\times\overline{x-OA6}=\tau,$$

$$\tau\times\overline{x-\frac{+OA2+OA7}{2}}=\upsilon,\ \tau\times\overline{x-OA2}\times\overline{x-OA7}=\varphi,$$

$$\varphi\times\overline{x-\frac{+OA+OA8}{2}}=\chi,\ \varphi\times\overline{x-OA}\times\overline{x-OA8}=\psi,$$

Et erit $k+\pi l+\rho m+\sigma n+\tau o+\upsilon p+\varphi q+\chi r+\psi s=PQ$.

P R O P. V.

Datis aliquot terminis ſeriei cujuſcunque ad data intervalla diſpoſitis, invenire terminum quemvis intermedium quamproxime.

Ad rectam poſitione datam erigantur termini dati in dato angulo, interpoſitis datis intervallis, & per eorum puncta extima, per Propoſitiones præcedentes, ducatur linea Curva generis Parabolici. Hæc enim continget terminos omnes intermedios per ſeriem totam.

PROP.

PROP. VI.

Figuram quamcunque Curvilineam quadrare quamproxime, cujus Ordinatæ aliquot inveniri possunt.

Per terminos Ordinatarum ducatur linea Curva generis Parabolici ope Propositionum præcedentium. Hæc enim figuram terminabit quæ semper quadrari potest, et cujus Area æquabitur Areæ figuræ propositæ quamproxime.

SCHOLIUM.

Utiles sunt hæ Propositiones ad Tabulas construendas per interpolationem Serierum, ut & ad solutiones Problematum quæ a quadraturis Curvarum dependent, præsertim si Ordinatarum intervalla & parva sint & æqualia inter se, & Regulæ computentur, & in usum reserventur pro dato quocunque numero Ordinatarum. Ut si quatuor sint Ordinatæ ad æqualia intervalla sitæ, sit A summa primæ & quartæ, B summa secundæ & tertiæ, & R intervallum inter primam & quartam, & Ordinata nova in medio omnium erit $\frac{9B-A}{16}$, & Area tota inter primam & quartam erit $\frac{A+3B}{8}R$.

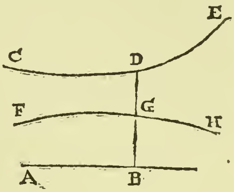
Et nota quod ubi Ordinatæ stant ad æquales ab invicem distantias, sumendo summas Ordinatarum quæ ab Ordinata media hinc inde æqualiter distant, & duplum Ordinatæ mediæ, componitur Curva nova cujus Area per pauciores Ordinatas determinatur, & æqualis est Areæ Curvæ prioris quam invenire oportuit. Quinetiam si pro Ordinatis novis sumantur summa Ordinatæ primæ & secundæ, et summa tertiæ & quartæ, et summa quintæ & sextæ, & sic deinceps; vel si sumantur summa trium primarum Ordinatarum, & summa trium proximarum, & summa trium quæ sunt deinceps; vel si sumantur summæ quaternarum Ordinatarum, vel summæ quinarum: Area Curvæ novæ æqualis erit Areæ Curvæ primo propositæ. Et sic habitis Curvæ quadrandæ Ordinatis quotcunque quadratura ejus ad quadraturam Curvæ alterius per pauciores Ordinatas reducetur.

Per

METHODUS DIFFERENTIALIS. 101

Per data vero puncta quotcunque non ſolum Curvæ lineæ generis *Parabolici*, ſed etiam Curvæ aliæ innumeræ diverſorum generum duci poſſunt.

Sunto CDE, FGH Curvæ duæ Abſciſſam habentes communem AB, et Ordinatas in eadem recta jacentes BD, BG; & relatio inter has Ordinatas definiatur per æquationem quamcunque. Dentur puncta quotcunque per quæ Curva CDE tranſire debet, & per æquationem illam dabuntur puncta totidem nova per quæ Curva FGH tranſibit. Per Propoſitiones ſuperiores deſcribatur Curva FGH generis *Parabolici* quæ per puncta illa omnia nova tranſeat, & per æquationem eandem dabitur Curva CDE quæ per puncta omnia primo data tranſibit.



F I N I S.

I.

METHODUS DIFFERENTIALIS.

(*Translation.*)

Prop. I.

If the abscissa of a curve consist of a given quantity A *and an indeterminate quantity* x, *and if the ordinate consist of any number of quantities* b, c, d, e, *&c., multiplied respectively into a corresponding number of terms of the* G. P. x, x^2, x^3, x^4, *&c., and if ordinates be erected at as many points of the abscissa; then the first differences of the ordinates are divisible by their intervals; and the differences of the differences so divided are divisible by the intervals between alternate ordinates; and the differences of these differences so divided are divisible by the intervals between every third ordinate, and so on indefinitely.*

Thus if given quantities p, q, r, s, t, &c., be substituted in succession for the indeterminate portion of the abscissa, and ordinates α, β, γ, δ, ϵ, &c., be erected at the extremities of the abscissæ so determined; the abscissæ and the ordinates, and the differences of ordinates divided by the differences of abscissæ (which are in fact the intervals of the ordinates), and the differences of the quotients divided by the differences of alternate ordinates, and so on, are shewn in the following table. (See p. 86.)

Prop. II.

Making the same suppositions and assuming the number of terms b, c, d, e, *&c., to be finite, the last quotient will be equal to the last of the terms* b, c, d, e, *&c., and the remaining terms will be found by means of the remaining quotients; and, when these terms are known, a parabolic curve is determined which passes through the extremities of all the ordinates.*

Thus, in the preceding table, the last quotient σ is equal to the last term e; and the product of this term by the known sum $p+q+r+s$ when subtracted from the quotient ξ leaves as remainder d, the last term but one. The quantities $d(p+q+r) + e(p^2+pq+q^2+pr+qr+r^2)$ which are then known, being deducted from the quotient λ, give the term c. The quantities $c(p+q) + d(p^2+pq+q^2) + e(p^3+p^2q+pq^2+q^3)$, which are then known, being deducted from the quotient ζ, leave the term b. By a similar calculation, other terms, if any, would be obtainable by means of a corresponding series of quotients. Finally, the ascertained quantities $bp+cp^2+dp^3+ep^4$, when deducted from the first ordinate α, leave the first term a of the expression for the ordinate. And the quantity $a+bx+cx^2+dx^3+ex^4$, &c., is the ordinate of a parabolic curve which passes through the extremities of all the given ordinates, the abscissa being $\mathrm{A}+x$.

From these propositions those which follow are easily inferred.

PROP. III.

If a straight line A_1A_9 *be divided into any number of equal parts* A_1A_2, A_2A_3, A_3A_4, A_4A_5, *&c., and if parallel straight lines* A_1B_1, A_2B_2, A_3B_3, *&c., be erected at the points of division, it is required to find a parabolic curve which shall pass through the extremities of all these lines.*

(See figure on p. 88).

Find the first differences b_1, b_2, b_3, &c., of the ordinates A_1B_1, A_2B_2, A_3B_3, &c. ; the second differences c_1, c_2, c_3, &c. ; the third differences d_1, d_2, d_3, &c. ; and so on, up to the last difference and let that difference be called i.

Then beginning at the last difference, take the central differences in the alternate columns or orders of differences, and the arithmetic means between the two central differences in the remaining columns, proceeding in order up to the series of primary terms A_1B_1, A_2B_2, A_3B_3, A_4B_4, &c. Call the terms extracted k, l, m, n, o, p, q, r, s, &c. ; the last of these symbols representing the last difference ; the last but one, the mean between the two differences in the last column but one, the last but two, the central difference of the three differences in the last column but two, and so on to the first of the symbols, which will represent either the central ordinate of the series A_1B_1, A_2B_2, A_3B_3, &c., or the mean between the two central ordinates, the former happening when the number of the ordinates is odd, and the latter when the number is even.

CASE I.

In the former case let A_5B_5 be the central ordinate, that is put

$$A_5B_5 = k, \quad \frac{b_4 + b_5}{2} = l, \quad c_4 = m, \quad \frac{d_3 + d_4}{2} = n,$$

$$e_3 = o, \quad \frac{f_2 + f_3}{2} = p, \quad g_2 = q, \quad \frac{h_1 + h_2}{2} = r, \quad i = s.$$

Erect the ordinate PQ and let $AP = x$. Now multiply continuously into one another the terms of the progression,

$$1,\ x,\ \frac{x}{2},\ \frac{x^2-1}{3x},\ \frac{x}{4},\ \frac{x^2-4}{5x},\ \frac{x}{6},\ \frac{x^2-9}{7x},\ \frac{x}{8},\ \frac{x^2-16}{9x},\ \frac{x}{10},\ \frac{x^2-25}{11x},\ \frac{x}{12},\ \frac{x^2-36}{13x},$$

&c., the resulting terms being

$$1,\ x,\ \frac{x^2}{2},\ \frac{x^3-x}{6},\ \frac{x^4-x^2}{24},\ \frac{x^5-5x^3+4x}{120},\ \frac{x^6-5x^4+4x^2}{720},$$

$$\frac{x^7-14x^5+49x^3-36x}{5040},\ \text{\&c.}$$

Then if these terms be respectively multiplied into the terms of the series k, l, m, n, o, p, &c., the sum of the products, namely

$$k + x \, . \, l + \frac{x^2}{2} \cdot m + \frac{x^3 - x}{6} \cdot n + \frac{x^4 - x^2}{24} \cdot o + \frac{x^5 - 5x^3 + 4x}{120} \cdot p + \&c.$$

will be the length of the ordinate PQ.

CASE II.

In the latter case let A_4B_4 and A_5B_5 be the two central ordinates, that is, put

$$\frac{A_4B_4 + A_5B_5}{2} = k, \; b_4 = l, \; \frac{c_3 + c_4}{2} = m, \; d_3 = n,$$

$$\frac{e_2 + e_3}{2} = o, \; f_2 = p, \; \frac{g_1 + g_2}{2} = q, \; \& \; h = r.$$

Erecting the ordinate PQ take the middle point O of A_4A_5, and call $OP = x$.

Now multiply continuously into one another the terms of the progression

$$1, \; x, \; \frac{x^2 - \frac{1}{4}}{2x}, \; \frac{x}{3}, \; \frac{x^2 - \frac{9}{4}}{4x}, \; \frac{x}{5}, \; \frac{x^2 - \frac{25}{4}}{6x}, \; \frac{x}{7}, \; \frac{x^2 - \frac{49}{4}}{8x}, \; \&c.$$

the resulting terms being

$$1, \; x, \; \frac{4x^2 - 1}{8}, \; \frac{4x^3 - x}{24}, \; \frac{16x^4 - 40x^2 + 9}{384}, \; \&c.$$

Then if these terms be respectively multiplied into the terms of the series k, l, m, n, o, p, q, &c., the sum of the products, namely,

$$k + x . l + \frac{4x^2 - 1}{8} \cdot m + \frac{4x^3 - x}{24} \cdot n + \frac{16x^4 - 40x^2 + 9}{384} \cdot o + \&c.$$

will be the length of the ordinate PQ.

It is to be noted that each of the intervals A_1A_2, A_2A_3, A_3A_4, &c., is here assumed to be unity; also that the differences are to be obtained by deducting the lower quantities from the upper, A_2B_2 from A_1B_1, A_3B_3 from A_2B_2, b_2 from b_1, &c., so that $A_1B_1 - A_2B_2 = b_1$, $A_2B_2 - A_3B_3 = b_2$, $b_1 - b_2 = c_1$, &c., and further that when any of the differences taken in this way turn out to be negative, effect must be given to the negative signs.

Prop. IV.

If a straight line be divided into any number of unequal parts A_1A_2, A_2A_3, A_3A_4, A_4A_5, *&c., and if parallel straight lines* A_1B_1, A_2B_2, A_3B_3, *&c., be erected at the points of division; it is required to find a parabolic curve which shall pass through the extremities of all the lines so erected.*

(See figure on p. 89.)

Let the given points be B_1, B_2, B_3, B_4, B_5, B_6, B_7, &c., and let fall ordinates B_1A_1, B_2A_2, &c., perpendicularly on the abscissa A_1A_7. Put

$$\frac{A_1B_1 - A_2B_2}{A_1A_2} = b_1, \quad \frac{A_2B_2 - A_3B_3}{A_2A_3} = b_2, \quad \frac{A_3B_3 - A_4B_4}{A_3A_4} = b_3,$$

$$\frac{A_4B_4 - A_5B_5}{A_4A_5} = b_4, \quad \frac{A_5B_5 - A_6B_6}{A_5A_6} = b_5, \quad \frac{A_6B_6 + A_7B_7}{A_6A_7} = b_6, \quad \frac{-A_7B_7 - A_8B_8}{A_7A_8} = b_7.$$

Thence derive

$$\frac{b_1 - b_2}{A_1A_3} = c_1, \qquad \frac{b_2 - b_3}{A_2A_4} = c_2, \qquad \frac{b_3 - b_4}{A_3A_5} = c_3, \text{ \&c.};$$

and then

$$\frac{c_1 - c_2}{A_1A_4} = d_1, \qquad \frac{c_2 - c_3}{A_2A_5} = d_2, \qquad \frac{c_3 - c_4}{A_3A_6} = d_3, \text{ \&c.};$$

and

$$\frac{d_1 - d_2}{A_1A_5} = e_1, \qquad \frac{d_2 - d_3}{A_2A_6} = e_2, \qquad \frac{d_3 - d_4}{A_3A_7} = e_3, \text{ \&c.};$$

the process being continued in the same way until the last difference is reached.

After the differences have been collected and divided by the intervals between the ordinates, the next step is to pick out the central terms in the alternate columns (or series, or lines), reckoning from the last difference, and the arithmetic means between the two central terms in the remaining columns, right up to the series of primary terms A_1B_1, A_2B_2, &c. Let the terms extracted be k, l, m, n, o, p, q, r, &c., of which the last symbol denotes the last difference; the last but one, the mean between the two differences in the last column but one; the last but two, the central difference of the three differences in the last column but two, &c. Then the first symbol k will represent the central ordinate if the number of ordinates is odd, or the arithmetic mean between the two central ordinates if their number is even.

CASE I.

In the former case let A_4B_4 be the central ordinate, that is, put $A_4B_4=k$, $\frac{b_3+b_4}{2}=l$, $c_3=m$, $\frac{d_2+d_3}{2}=n$, $e_2=o$, $\frac{f_1+f_2}{2}=p$, $g=q$. Having erected the ordinate PQ, take a fixed point O in the base A_1A_5 and call $OP=x$. Then multiply into one another in succession the terms of the progression,

$$1,\ x-OA_4,\ x-\frac{OA_3+OA_5}{2},\ \frac{(x-OA_3)(x-OA_5)}{x-\frac{1}{2}(OA_3+OA_5)},$$

$$x-\frac{OA_2+OA_6}{2},\ \&c.,$$

and take the resulting progression.

Or, what comes to the same thing, multiply into one another in succession the terms of the progression,

$$1,\ x-OA_4,\ (x-OA_3)(x-OA_5),\ (x-OA_2)(x-OA_6),$$

$$(x-OA_1)(x-OA_7),\ \&c.$$

Then multiply the resulting terms respectively into the terms of the progression,

$$1,\ x-\frac{OA_3+OA_5}{2},\ x-\frac{OA_2+OA_6}{2},\ x-\frac{OA_1+OA_7}{2},\ \&c.,$$

and intermediate terms will be obtained, the complete progression being

$$1,\ x-OA_4,\ x^2-\frac{OA_3+2OA_4+OA_5}{2}\cdot x+\frac{OA_3+OA_5}{2}\times OA_4,\ \&c.$$

Otherwise, let $OA_1=\alpha$, $OA_2=\beta$, $OA_3=\gamma$, $OA_4=\delta$, $OA_5=\epsilon$,

$$OA_6=\zeta,\ OA_7=\eta,\ \frac{OA_3+OA_5}{2}=\theta,\ \frac{OA_2+OA_6}{2}=\chi,\ \frac{OA_1+OA_7}{2}=\lambda.$$

Obtain terms by continuous multiplication from the progression

$$1,\ x-\delta,\ x-\gamma,\ x-\epsilon,\ x-\beta,\ x-\zeta,\ x-\alpha,\ x-\eta,\ \&c.,$$

and obtain intermediate terms by multiplying the results respectively by

$$1,\ x-\theta,\ x-\chi,\ x-\lambda,\ \&c.,$$

the whole series being

$$1,\ x-\delta,\ x^2-(\delta+\theta)x+\delta\theta,\ x^3-(\delta+2\theta)x^2+(\gamma\epsilon+2\delta\theta)x-\gamma\delta\epsilon,\ \&c.,$$

the terms of which are to be multiplied respectively by k, l, m, n, o, p, &c. Then the sum of the products, namely

$$k+(x-\delta).l+\{x^2-(\delta+\theta)x+\delta\theta\}.m+\&c.$$

will be the length of the ordinate PQ.

CASE II.

In the latter case let A_4B_4, A_5B_5, be the two middle ordinates, that is, put

$$\frac{A_4B_4+A_5B_5}{2}=k,\quad b_4=l,\quad \frac{c_3+c_4}{2}=m,\quad d_3=n,\quad \frac{e_2+e_3}{2}=o,\quad f_2=p,\quad \&c.$$

The co-efficients of the alternate terms are obtained by the continuous multiplication of the terms of the progression,

$$1,\ (x-OA_4)(x-OA_5),\ (x-OA_3)(x-OA_6),\ (x-OA_2)(x-OA_7),$$
$$(x-OA_1)(x-OA_8),\ \&c.,$$

and those of the remaining terms are obtained by multiplying the above co-efficients by the terms of the progression,

$$x-\frac{OA_4+OA_5}{2},\ x-\frac{OA_3+OA_6}{2},\ x-\frac{OA_2+OA_7}{2},\ x-\frac{OA_1+OA_8}{2}.$$

Then

$$k+\left(x-\frac{OA_4+OA_5}{2}\right).l+\{x^2-(OA_4+OA_5).x+OA_4\times OA_5\}.m+\&c.,$$

will be the ordinate PQ ;

$$\text{or,}\quad PQ=k+\left(x-\tfrac{1}{2}OA_4-\tfrac{1}{2}OA_5\right).l+(x-OA_4)(x-OA_5).m$$
$$+(x-OA_4)(x-OA_5)\left(x-\tfrac{1}{2}OA_5-\tfrac{1}{2}OA_6\right).n,\ \&c.$$

Thus putting $\quad x-\dfrac{OA_4+OA_5}{2}=\pi,\quad (x-OA_4)(x-OA_5)=\rho,$

$$\rho.\left(x-\frac{OA_3+OA_6}{2}\right)=\sigma,\quad \rho.(x-OA_3)(x-OA_6)=\tau,$$
$$\tau.\left(x-\frac{OA_2+OA_7}{2}\right)=\upsilon,\quad \tau.(x-OA_2)(x-OA_7)=\phi,$$
$$\phi.\left(x-\frac{OA_1+OA_8}{2}\right)=\chi,\quad \phi.(x-OA_1)(x-OA_8)=\psi.$$

the equation to the curve will be

$$k+\pi.l+\rho.m+\sigma.n+\tau.o+\upsilon.p+\phi.q+\chi.r+\psi.s=PQ.$$

Prop. V.

Certain terms out of a sequence of values being given, arranged at known intervals, it is required to find any intermediate term as closely as possible.

On a fixed straight line erect at a constant angle the given terms arranged at the given intervals; and let a parabolic curve be drawn through their extremities by means of the preceding propositions. This curve will pass through the extremities of all the intermediate terms.

Prop. VI.

To find the approximate area of any curve a number of whose ordinates can be ascertained.

Let a parabolic curve be drawn through the extremities of the ordinates by means of the preceding propositions. This will form the boundary of a figure whose area can always be ascertained, and its area will be approximately equal to the area required.

SCHOLIUM.

These propositions are useful for the construction of tables by the interpolation of series, as also for the solution of problems which depend on finding the areas of curves, especially if the intervals between the ordinates are small and equal to one another; and rules applicable to any given number of ordinates can be derived and recorded for reference. For example: If there are four ordinates at equal intervals, let A be the sum of the first and fourth, B the sum of the second and third, and R the interval between the first and fourth; then the central ordinate will be $\frac{9B - A}{16}$, and the area between the first and fourth ordinates will be $\frac{A + 3B}{8}R$.

Note also that when the ordinates stand at equal distances from one another, the sums of ordinates which are equally distant from the central ordinate, along with twice the central ordinate, supply data for a new curve the area of which is determined by means of a smaller number of ordinates and is equal to the area of the original curve. Moreover, if the sum of the first and second ordinates, the sum of the third and fourth, the sum of the fifth and sixth, and so on in succession are taken as new ordinates; or if the sum of the first three ordinates, the sum of the next three, and the sum of the succeeding three are taken; or if the sums of the ordinates are taken four at a time, or five at a time, the area of the new curve will be equal to that of the original curve. And in this way, when a number of ordinates are given of a curve whose area is required, the calculation of the area is reduced to that of the area of another curve by means of a smaller number of ordinates.

Through a given number of points not only parabolic curves but an infinity of other curves of different kinds can be drawn.

(See figure on page 93.)

Let CDE, FGH, be two curves having a common abscissa AB, and ordinates BD, BG, lying in the same straight line; and let the relation between the ordinates be defined by any equation whatever. Let any number of points be given through which the curve CDE is required to pass, and by that equation an equal number of points will be given through which the curve FGH will pass. By means of the foregoing propositions let a parabolic curve FGH be described passing through those points, and by the same equation a curve CDE will be given which will pass through all the points first given.

II.

LETTER ON CONSTRUCTION OF TABLES.

LETTER CIX.

W. Jones to Prof. Cotes.

LONDON, *Jan. 1st,* $17\frac{11}{12}$.

Dr. Sr.

I have sent you here inclos'd, the Coppy of a Letter, that I found among Mr. Collins's papers, from Sr. Is. Newton to one Mr. Smith; the contents thereof seem to have, in some measure, relation to what you are about, as being the application of the Doctrine of Differences to the making of Tables; and for that reason I thought it might be of use to you, so far as to see what has bin done already: I shew'd this to Sr. Isaac, he remembers yt. he apply'd it to all sorts of Tables, but has nothing by him more than what is printed: I have more papers of Mr Mercator's and others, upon this subject, tho, I think, none so material, as this. I shou'd be very glad to see what you have done of this kind all publish'd. And I must confess, that, unless you design a considerable large Volume, 'twere much better to put them into the Transactions; for that wou'd sufficiently preserve them from being lost, which is ye. common fate of small single Tracts; and at ye. same time save the trouble and expense of printing them, since the subject is too curious to expect any profit by it: and besides, now, as the R. Society having done themselves the honour of choosing you a Member, something from you cannot but be acceptable to them: Sr. Isaac himself expects those things of yours that I formerly mentioned to him as your promise.

I am, Sr. your much oblig'd
friend, & humble Servt.
W. JONES.

LETTER CIX. (bis).

Newton to J. Smith.

[Enclosed in Letter CIX.] [Copy.]

Trin. Coll. Cambridge, *May 8th*, 1675.

Sr.

I have consider'd ye. buisiness of computing Tables of Square, Cube, & Sq. Sqr. Roots; and ye. best way of p'forming it, yt. I can think of is yt. which follows:

If yu. wo'd compute a Table to 8 decimal places, let ye. roots of every hundredth number be extracted to ten decimal places, and then compute every tenth numbr. and afterwards every number by the following methods:

Tab. I.

$n-50$	$*o$	α	
	m	o	
$n-40$	op	β	
	m	π	
$n-30$	pq	γ	
	m	χ	
$n-20$	qr	δ	
	m	ρ	
$n-10$	rs	ϵ	
	m	σ	
n	st	ζ	$\frac{m}{10}$
	m	τ	
$n+10$	tv	η	
	m	υ	
$n+20$	vx	θ	
	m	ϕ	
$n+30$	xy	ι	
	m	ψ	
$n+40$	yz	κ	
	m	ω	
$n+50$	$z*$	λ	

Tab. II.

$n-6$	4E	
	5ϵ	
$n-5$	5E. F5	
	$\zeta 4$	
$n-4$	F4	
	$\zeta 3$	
$n-3$	F3	
	$\zeta 2$	
$n-2$	F2	
	$\zeta 1$	
$n-1$	F1	
	ζ	
n	F	$\frac{st}{100}$
	1ζ	
$n+1$	1F	
	2ζ	
$n+2$	2F	
	3ζ	
$n+3$	3F	
	4ζ	
$n+4$	4F	
	5ζ	
$n+5$	5F. G5	
	$\eta 4$	
$n+6$	G4	
	$\eta 3$	
$n+7$	G3	
	$\eta 2$	
$n+8$	G2	
	$\eta 1$	
$n+9$	G1	
	η	
$n+10$	G	$\frac{tv}{100}$
	1η	
$n+11$	1G	
	2η	
$n+12$	2G	

In the First Table.

Let n signify every 100th numbr. & F its root, whether Square, Cube, or Sq. Square; and $n-50$, $n-40$, $n-30$, &c., every tenth numbr.; and A, B, C, D, &c., their roots; and o, p, q, r, &c., the differences of these roots; and op, pq, qr, &c., their second differences, (that is op, the diff. of o & p, pq the diff. of p & q, &c.) and m their third difference, that is ye. common difference of *o & op, op & pq, pq & qr, &c.

Further, let α, β, γ, δ, &c., signify ye. the differences of these Roots from those next less, namely, α the difference of ye. root of $n-50$ & ye. like root of $n-51$, β, the diff. of ye. roots $n-40$ & $n-41$, ζ the diff. of ye. roots of n & $n-1$, η the diff. of ye. roots of $n+10$ & $n+9$, &c. And let o, π, χ, ρ, &c., signify the diff. of α, β, γ, δ, &c. And $\frac{m}{10}$ the common diff. of o, π, χ, ρ, &c.

In the Second Table.

Let $n-6$, $n-5$, $n-4$, $n-3$ &c. signify ye. single numbers,

4E, 5E, or F5, F4, F3 &c. their roots,

5ϵ, $\zeta 4$, $\zeta 3$, $\zeta 2$ &c. the diff. of those roots;

$\frac{st}{100}$ the common diff. of those differences for ye. ten numbers between $n-5$ & $n+5$.

And so for ye. ten numbers between $n+5$ & $n+15$; let G5, G4, G3, &c. signify ye. roots; $\eta 4$, $\eta 3$, $\eta 2$ &c. their first differences, and $\frac{tv}{100}$ their second differences; and the like for every denarie between $n-50$ & $n+50$.

This explication of the Tables being p'mis'd, you may compute them thus;

$$\text{Out of } n, \text{ extract F ye.} \left\{\begin{matrix}\text{Square}\\ \text{Cube}\\ \text{Sq. Sq.}\end{matrix}\right\} \text{Root, make} \left\{\begin{matrix} \frac{10F}{2n}=\omega, & \frac{10\omega}{2n}=st, & \frac{30st}{2n}=m \\ \frac{10F}{3n}=\omega, & \frac{20\omega}{3n}=st, & \frac{50st}{3n}=m \\ \frac{10F}{4n}=\omega, & \frac{30\omega}{4n}=st, & \frac{70st}{4n}=m \end{matrix}\right.$$

$$\omega+\frac{1}{2}st+\frac{1}{6}m=s,\quad \frac{\omega}{10}+\frac{\frac{1}{2}st}{100}+\frac{\text{‘}m\text{’}}{6000}\dagger=\zeta, \text{ and } \frac{st}{10}+\frac{55m}{1000}=\sigma$$

And these quantities F, st, m, s, ζ, & σ, being thus found, ye. rest are given by Additn. & Subduct.

† *Note by Editor* (J. EDLESTON).—I have added the $\frac{\text{‘}m\text{’}}{6000}$. I have also corrected some other errors of transcription.

Note by D. C. F.—The quantity $\frac{m}{6000}$ is so small that Newton properly omitted it.

For $st+m=rs$, $rs+m=qr$, &c. $st+m=tv$, $tv+m=vx$, &c.

Again $s+rs=r$, $r+rq=q$, &c. $s-st=t$, $t-tv=v$, &c.

And $F-s=E$, $E-r=D$, &c. $F+t=G$, $G+v=H$, &c.

Further

$$\sigma+\frac{m}{10}=\rho,\ \rho+\frac{m}{10}=\chi,\ \&c.\quad \sigma-\frac{m}{10}=\tau,\ \tau-\frac{m}{10}=v,\ \&c.$$

Lastly $\zeta+\sigma=\epsilon$, $\epsilon+\rho=\delta$, &c. $\zeta-\tau=\eta$, $\eta-v=\theta$, &c.

These quantities being thus computed in the first Table, to every 10th number, the roots may be computed in ye. 2d Table to every numbr. by Addition and Subduction only ;

For $$\zeta+\frac{st}{100}=\zeta1,\ \zeta1+\frac{st}{100}=\zeta2,\ \&c.$$

$$\zeta-\frac{st}{100}=1\zeta,\ 1\zeta-\frac{st}{100}=2\zeta,\ \&c.$$

Again $F-\zeta=F1$, $F1-\zeta1=F2$, &c.

$F+1\zeta=1F$, $1F+2\zeta=2F$, &c.

Thus you must proceed to five Figures on either hand, and then do the like in the next ten Figures, saying

$$\eta+\frac{tv}{100}=\eta1,\ \eta1+\frac{tv}{100}=\eta2,\ \&c.$$

And the like for every Denarie between $n-50$ & $n+50$.

In these Computations, Note, 1st.—That they must be done everywhere to 10 or 11 decimal places, if you will have a Table of Roots exact to 8 of these places.

2dly.—If 5F & G5, the roots of $n+5$ found two ways agree to 8 decimal places, it argues the whole works from which they were derived to be true. And so of ye. roots of $n+15$, $n+25$, $n-5$, &c. And also of ye. Terms A, $^*o$, & a; L, z^*, & λ where two works meet. Let this therefore be ye. Proof of ye. work.

This Sr. is wt. has occurred to me about your design, which I hope will do your business, the whole work being p'form'd by Addit. & Subduct: excepting yt. in ye. computation of every 100th number, there is required ye. Extraction of one root, & three divisions to find F, ω, st, & m.

Sr. I am,

Your humble Servt.

IS. NEWTON.

[*Note appended by Editor,* J. EDLESTON.]

The person to whom this letter is written may be conjectured to be "John Smith, Philo-Accomptant", author of Stereometrie, Lond. 1673. (He must not

be confounded with Cotes's uncle). In the Macclesfield Correspondence, II, 370–374, there are two other letters on the extraction of roots from Newton to this same person (not to Collins, as there printed) dated July 24 and Aug. 27, 1675, in the former of which he refers to the method given in the foregoing letter. Mr. J. Smith seems to have had a design of constructing Tables of Square, Cube and Biquadr. Roots, and consulted Newton as to the best mode of computing them. The Tables, if ever made, do not appear to have been published. The earliest Tables of Roots are Briggs's MS. Tables of the Square Roots of Numbers up to 1000 mentioned in Mayne's Merchant's Companion (London, 1674), p. 80.

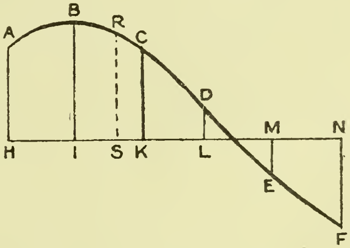
III.

PRINCIPIA. BOOK III. LEMMA V.

(*Translation.*)

To find a parabolic curve which shall pass through any given points:

b		b_2		b_3		b_4		b_5
	c		c_2		c_3		c_4	
		d		d_2		d_3		
			e		e_2			
				f				



Let the points be A, B, C, D, E, F, &c., and from them let fall perpendiculars AH, BI, CK, DL, EM, FN, on any given straight line HN.

Case I.—If the intervals HI, IK, KL, &c., between the points H, I, K, L, M, N, are equal, find the first differences b, b_2, b_3, b_4, b_5, &c., of the perpendiculars; the second differences c, c_2, c_3, c_4, &c.; the third differences d, d_2, d_3, &c., so that $\text{AH} - \text{BI} = b$, $\text{BI} - \text{CK} = b_2$, $\text{CK} - \text{DL} = b_3$, $\text{DL} + \text{EM} = b_4$, $-\text{EM} + \text{FM} = b_5$, &c.; $b - b_2 = c$, &c.; and so on, to the last difference, which in this case is f. Then having erected any perpendicular RS, which shall be an ordinate to the required curve, in order to find its length put each of the intervals HI, IK, KL, LM, equal to unity and let $\text{AH} = a$, $-\text{HS} = p$, $\frac{1}{2}p \times (-\text{IS}) = q$, $\frac{1}{3}q \times \text{SK} = r$, $\frac{1}{4}r \times \text{SL} = s$, $\frac{1}{5}s \times \text{SM} = t$, proceeding in the same way to the last but one of the perpendiculars ME, and prefixing the negative sign to the terms HS, IS, &c., which lie on the same side of the point S as A, and the positive signs to the terms SK, SL, &c., which lie on the other side of the point S. Then, observing the correct signs, RS will be $= a + bp + cq + dr + es + ft$, &c.

Case II.—But if the intervals HI, IK, &c., between the points H, I, K, L are unequal, find b, b_2, b_3, b_4, b_5, being the first differences of the perpendiculars AH, BI, CK, &c., divided by the

intervals between the perpendiculars; c, c_2, c_3, c_4, &c., the second differences divided by the intervals between the alternate perpendiculars; d, d_2, d_3, &c., the third differences divided by the intervals between every third perpendicular; e, e_2, &c., the fourth differences divided by the intervals between every fourth perpendicular, and so on; so that $b=\frac{\text{AH}-\text{BI}}{\text{HI}}$, $b_2=\frac{\text{BI}-\text{CK}}{\text{IK}}$, $b_3=\frac{\text{CK}-\text{DL}}{\text{KL}}$, &c.; $c=\frac{b-b_2}{\text{HK}}$, $c_2=\frac{b_2-b}{\text{IL}}$, $c_3=\frac{b_3-b_4}{\text{KM}}$, &c.; $d=\frac{c-c_2}{\text{HL}}$, $d_2=\frac{c_2-c_3}{\text{IM}}$, &c. The differences having been ascertained, put $\text{AH}=a$, $-\text{HS}=p$, $p\times(-\text{IS})=q$, $q\times\text{SK}=r$, $r\times\text{SL}=s$, $s\times\text{SM}=t$; proceeding, it will be understood, to the last perpendicular but one, ME; then the ordinate RS will be $=a+bp+cq+dr+es+ft$, &c.

Corollary.—Hence the areas of all curves can be ascertained approximately. For if several points are found of the curve whose area is required, and a parabolic curve be supposed drawn through these points, its area will be approximately the same as that of the given curve. But the area of the parabolic curve can always be found geometrically by methods that are very well known.

Notes on some recent developments of Pension Problems in America. By W. J. H. Whittall, F.I.A., F.A.S.

Introductory.

SOME interesting reports have come to hand recently from America showing that pension questions, and more particularly the proper methods of financial provision, are receiving a great deal of attention throughout the United States and Canada. A thorough exploration of these sources of information by actuarial enquirers is clearly desirable in the public interest, especially in view of the enormous accretions that are being made to the pension liabilities of this country as a consequence of the war, both directly on the military side and indirectly through the great increase in the numbers of the bureaucracy and the rising scales of all salaries and wages.

For instance, the Prime Minister on a recent Saturday afternoon promised the Metropolitan Police, who were on strike, an increase of pensionable pay of 13*s.* a week. The additional liability for pensions alone thus placed on the taxpayer must represent a present capital value of some

millions of pounds; but no information has been given on this point, nor is it known even whether a calculation of the cost was made before the liability was incurred. A still more disturbing incident is the issue of an official statement by the President of the Board of Education (Cd. 9141), that he proposes, obviously with the consent of the Treasury, to introduce a Bill which "will bring within one comprehensive "system of State pensions, on a non-contributory basis, the "certificated teachers, the uncertificated teachers, and the "teachers of special subjects in elementary schools, and the "teachers in all other schools aided by the Board of "Education, including those training colleges which are not "departments of universities." The scale of benefits will closely resemble that of the Civil Service Superannuation Act of 1909, and, speaking generally, all existing staffs, male and female, are to be included in respect of past service. As I understand the statement, the contributory scheme for all secondary schools recommended by a departmental committee (Cd. 7365), to be effected by means of insurance policies, will now be superseded. As the total salary list in the aided secondary schools and training colleges is over £2,500,000, I should not be surprised if in their case the total present value of Mr. Fisher's proposed benefits amounted to as much as £8,000,000 or £10,000,000 for the existing staffs. As regards the elementary schools, I have not salary statistics at hand, but Sir George Hardy and Mr. D. C. Fraser found over 100,000 teachers in service in 1912, and an estimated present salary list of £15,000,000 would be not unreasonable. The present value of the pensions in their case might thus amount to £50,000,000 or more. But this would be for existing staffs only. If the service be regarded as a going concern, and the cost be reckoned of replacing outgoing members, the present value would be largely increased. In the case of some of the State services in America, it has been found that this consideration has more than doubled the actuarial estimates of the present capital cost. I only mention these rough guesses to show what large sums are in question, and the urgent need of proper information.

It will probably be found that a great deal is taking place in America which has a distinct bearing on the numerous schemes now being introduced or contemplated in this country. By way of a beginning, I propose to give a short account of

the Carnegie Foundation for the Advancement of Teaching; and to supplement this later by drawing attention to a few of the recent developments in the field of municipal and state pensions.

The Carnegie Foundation for the Advancement of Teaching.

In 1905 Mr. Andrew Carnegie set aside £2,000,000 in 5 per-cent Steel Bonds (the dollar being reckoned at 4*s.*) to benefit the teachers engaged in higher education in the United States and Canada, and the "Carnegie Foundation "for the Advancement of Teaching" was established. Its Twelfth Report, written by Dr. H. S. Pritchett, the President, and issued from 576, Fifth Avenue, New York City, contains a good deal of information apart from their own experience. Except where otherwise stated, my facts are taken from this Report. For the following historical summary of the proceedings of the Foundation I am partly indebted to a memorandum prepared by Mr. James Robb, the Secretary of the Carnegie Universities Trust in Edinburgh.

The revenue of £100,000, in the words of the Certificate of Incorporation, was mainly "to provide retiring pensions, "without regard to race, sex, creed, or colour, for the teachers "of universities, colleges, and technical schools in the United "States, the Dominion of Canada, and Newfoundland, who, "by reason of long and meritorious service in these "institutions, shall be deemed by the Board of Directors to "be entitled to the assistance and aid of this Corporation, or "who, by reason of old age or disability, may be prevented "from continuing in the active work of their profession; to "provide for the care and maintenance of the widows and "families of the said teachers. . ." Mr. Carnegie, in his letter to the trustees, of date 16 April 1905, gave his reasons for the Foundation. He had reached the conclusion that the least rewarded of all the professions was that of the teacher in the higher educational institutions of America, that able men hesitated to adopt teaching as a career, and many old professors, whose places should be occupied by younger men, could not be retired, and that expert calculation showed the revenue would be ample for the purpose he had in view.

There were to be no contractual relations and no legal promises as to future pensions. At the same time, in order

to preserve the self-respect of the recipients, it was considered necessary to avoid any eleemosynary taint, and the pensions were to be regarded as fairly earned by the past service. After mature deliberation and consultation with actuaries, administrators, teachers, and publicists, the trustees adopted a system of retiring allowances which was non-contributory and stipendiary in character, namely, one-half of the average salary for the last five years plus £80; they admitted a selected list of institutions to the permanent privileges of the pension endowment, believing that it was better to establish a fair retiring allowance system in a limited number of colleges than a very poor system in a large number; they decided to award these pensions at the minimum age of 65, and to give a smaller pension after 25 years of service as a professor, on the ground of disability; and they extended the privileges of the Foundation to instructors and to the widows of teachers.

The experience acquired in the working of this system for a few years banished some delusions, such as the expectation on the part of many teachers that they would be able to retire in the fifties and spend the rest of their lives in literary art or scientific research, the anticipation of college presidents that inefficient men could be disposed of by a pension, as well as the expectation of the trustees that scientific research could be stimulated by a judicious use of retiring allowances. It was also soon found out that the revenue, which expert calculation estimated would be ample for the purpose, was to be quite insufficient to meet the demands made upon it by eligible applicants. It had been stated in the First Annual Report that the average pension paid by the Trust was then £290, and was expected to be less in future. At the close of 1912, six years after the Trust was initiated, the general average had increased to fully £335, and was likely to increase further. It was also estimated that the pension system could be maintained at a cost of somewhere between 7 and 10 per-cent of the salary roll, and that an income of £100,000 would maintain such a system for a group of between 3,000 and 4,000 teachers. At 30 September 1912 the amount distributed annually had already grown to £114,085 for 429 retiring allowances, and 90 widows' pensions in all, and the cost then amounted in general to about 4 per-cent of the salary roll of the accepted institutions. Finally, on 1 April

1917 there were 6,593 teachers, including professors and instructors, in the associated colleges and universities (now 73 in number), and at 30 June, when the general endowment of the Foundation stood at almost £2,900,000 the total number of allowances in force was 480, and the cost £147,872. As regards the future, the estimate now is that for nearly the whole of the coming fifty years the load upon the Foundation, arising from the continuation of the present rule, even if limited to existing staffs, would exceed the present income of the Foundation. It was assumed that the income for the purposes of pension payments would be approximately £150,000 per annum, and it was calculated that in forty-five years there would be a deficit of some millions. Thus the work, begun with the best intentions, and with the utmost desire to serve the teaching body, resulted in conclusions which were quite different from the assumptions upon which it started.

Clearly the whole problem of pensions had to be faced by the Trust, and a new basis found if possible. Very careful consideration was given to the problem in all its bearings from time to time. On 15 April 1916 Dr. H. S. Pritchett, the President, by direction of the trustees, issued for general distribution his pamphlet entitled, "A Comprehensive Plan "of Insurance and Annuities for College Teachers." This plan claimed to be based upon the results of a study of the personal statements of some 4,000 college teachers who had furnished full information as to their experience in the effort to protect their families against the risks incident to their calling. The conclusion of the enquiry was that a contributory system in which both teacher and college joined, and which was so constructed that it would not restrict migration from one college to another, was the only system of retiring annuities which was at once socially wise, economically sound, and permanently secure.

Criticism of the pamphlet and the advice of all interested—teachers, actuaries, and publicists—were invited. Voluminous correspondence followed, and it was then resolved that the plan be referred to a Joint Commission consisting of six trustees of the Foundation, two representatives of the American Association of University Professors, and one representative each from the Association of American Universities, the National Association of State Universities,

and the Association of American Colleges, assisted by an actuary. This body of eleven individuals was thus representative of the leading educational authorities of America. On 27 April 1917 it agreed upon a comprehensive plan of insurance and annuities which was recommended to the trustees, and to a description of which I will return presently.

The difficulties in which the Carnegie Foundation had found itself involved are very much those which would have been foreseen by competent actuarial opinion. Lessons derived from actual experience, however, are often more readily received than any results of *a priori* reasoning, however competent. It may, therefore, be worth while to quote at length, and in Dr. Pritchett's own words, the conclusions arrived at by the Trustees of the Foundation, as the result of their 12 years' experience of "hit or miss" methods of granting pensions.

The past year has been a notable one in the history of the Carnegie Foundation. It has marked the culmination of some twelve years of study and investigation.

When the Carnegie Foundation was established, the founder and those associated with him had no thought of any other form of pension system than that of a teacher's pension paid entirely from an endowment provided for that purpose, and without the participation of the teacher. The concern of the trustees and of the founder was so to introduce such a pension to the colleges and teachers of the United States, Canada, and Newfoundland, that it might not seem a gratuity—that it might come as a right, not as a favour . . . . . .

The experience of twelve years and the examination of the pension problem in all countries has led the trustees to no different conclusion as to the need for making any pension a matter of right, not a matter of favour. The exhaustive study made in these intervening years has shown clearly that no system of free pensions can be devised which will not in the end affect the teacher's pay. The experience of the world has also now accepted the economic truth that the members of any group in the body politic receive their best service at the hands of society when the machinery is provided by which they may attain therein economic independence, rather than have the risk of dependence lifted from them by free gift. It is also clear that the opportunity to protect themselves against dependence should be open to the great body of teachers, not to a selected minority.

Under the pressure of these conclusions the officers and trustees of the Carnegie Foundation were faced with two duties: first, to carry out fairly and to the best of their ability the obligations assumed in the associated institutions; and secondly, to establish

as quickly as possible a system of benefits open to the great body of teachers in the three English-speaking countries of North America, upon terms which should be permanent, economically sound, and within the reach of the teacher and of his college . . . . . The second question—namely, what are the sound bases for the protection of the lives of teachers against dependence?—has been difficult to deal with, but the conclusions to which all authorities have come have, on the whole, been more generally accepted. The result of the twelve years' work of the Foundation has been epoch-making in this respect. The studies brought together from all sources have demonstrated that the problem of insurance and the problem of the annuity cannot be financially separated. This study has made clear the fact that a contributory system of annuities is the only one which society can permanently support, and under which the teacher shall be sure of his protection. Finally, these studies have made clear that the establishment of a co-operation for the common protection of college teachers in the United States, Canada, and Newfoundland, would mean in a generation a security for the profession of the teacher, and a solidarity of feeling which nothing else can bring about.

While the work of these years has been difficult, while questions have necessarily been raised which brought criticism and alarm, the results of this work have nevertheless been the greatest contribution to the profession of the teacher which the Carnegie Foundation could possibly accomplish. Nothing will count so much for the future as to bring about a form of co-operation under which each man who enters the teacher's profession may be able to enter into a contract whereby he and his family shall be protected against dependence, and at a cost within his ability to pay. It is to be regretted that this knowledge was not available when the Foundation began. The one thing which such an agency can do is to proceed sincerely and courageously, once the information is in its hands. This is what the Carnegie Foundation has undertaken to do. It is encouraging to find from the widespread correspondence which comes to the Foundation, that the great body of teachers have realized the effort which has been made, have sympathized with the difficulties to be overcome, and have accepted in the best spirit the results to which the studies have led.

The past year of the Foundation—marking as it does the acceptance of the fundamental principles worked out in the ten years past, committing the Foundation definitely to the principles of the contributory pension, to a plan of insurance and annuities which mutually support each other, and to an organization open to the great body of college teachers of the whole Continent—is the combination of twelve years of work, begun with the best intentions and with the highest desires to serve the teaching body, but resting in the end upon conclusions quite different from the assumptions upon which the work started. In this day of world trial, the Carnegie Foundation has sought to deal sincerely both with its obligations and its mistakes.

It will now be interesting to revert to the detailed conclusions of the Joint Commission to which reference has been made. In the first instance, it had tables submitted by actuaries showing that the probable future "load" of the pensions exceeded the income in prospect. The following extract is made from these statements as illustrating a point discussed a good deal in this country in reference to State pensions, namely, the impossibility of accepting the relation of the non-effective votes to the effective votes—that is, of current pensions to current salaries—as any guide to future commitments where the numbers in service are increasing rapidly. It was found that the Foundation had cognizance of 2,898 members of existing male staffs between the ages of 35 and 64. If their age distribution had been that of a normal population (McClintock's Table) it would have been as set out in column 2. Their actual age distribution was found to be as set out in column 3.

Age period (1)	Estimated Distribution of a Stationary Population (2)	Actual Distribution (3)
35–39	570	904
40–44	540	715
45–49	509	550
50–54	473	359
55–59	430	211
60–64	376	159
Total	2,898	2,898

It is, of course, obvious that the numbers approaching age 65 at the present time are no guide to the number of future pensioners.

After consideration of this question, the Joint Commission frankly agreed that the obligations of the Foundation need not be regarded as extending in any case beyond existing teachers; and while hoping that future and further subsidies from extraneous sources might enable it to bear the heavy but limited future load as regards existing teachers, it recognized that the ever-increasing burden which obviously would accrue with future appointments could not be borne without contributions. The Joint Commission thereupon, with typical American thoroughness and effort to get down to

logical foundations, voted a series of fundamental principles for a sound pension system which may be given in full.

I.

1. The function of a pension system is to secure to the individual who participates in it protection against the risk of dependence due to old age or to disability.

2. The obligation to secure this protection for himself and for his family rests first upon the individual. This is one of the primary obligations of the existing social order. Society has done its best for the individual when it provides the machinery by which he may obtain this protection at a cost within his reasonable ability to pay.

3. Men either on salary or on wages, are, in the economic sense, employees. The employer, whether a government, a corporation, or an individual, has a direct financial interest in the establishment of some pension system which shall enable old or disabled employees to retire under satisfactory conditions. In addition, society demands to-day that the employer assume some part in the moral and social betterment of his employees. The obligation of the employer to co-operate in sustaining a pension system is primarily a financial one, and in the second place, a moral one.

4. A pension system designed for any group of industrial or vocational workers should rest upon the co-operation of employee and employer.

5. Teachers' pensions should be stipendiary in character, amounting to a fair proportion of the active pay.

II.

1. In actuarial terms a pension is a deferred annuity upon the life of one or more individuals, payable upon the fulfilment of certain conditions.

2. In order that an individual participating in a pension system may be assured of his annuity when due, one condition is indispensable: There must be set aside, year by year, the reserve necessary, with its accumulated interest, to provide the annuity at the age agreed upon. On no other conditions can the participator obtain a satisfactory contract. The man of thirty who participates in a pension plan under which he expects an annuity thirty-five or forty years in the future will take some risk of disappointment in accepting any arrangement less secured than a contractual one.

3. A pension system conducted upon the actuarial basis of setting aside, year by year, the necessary reserve is the only pension system whose cost can be accurately estimated in advance.

4. A method by which a pension is paid for in advance in annual or monthly instalments is the most practical plan which can be devised for purchasing a deferred annuity, provided that the contributions begin early in the employee's career, and provided also that the contributions paid in year by year receive the benefit of the current interest for safe investments.

5. As a matter of practical administration, a pension system should apply to a group whose members live under comparable financial and economic conditions. To attain its full purpose, participation in the pension system to the extent of an agreed minimum should form a condition of entering the service or employment the members of which are co-operating in the pension system.

As to the machinery by which these principles could be put in operation, the Commission recommended that a new Teachers' Insurance Association should be formed under the laws of the State of New York. The joint contributions of employer and employed would be paid to this Association in exchange for:

(1) *Term Insurances,* for family provision, to last until age 65, and of "flexible" amounts to suit varying circumstances, and

(2) *Deferred Annuities,* to begin at age 65, with return of premiums and interest in the event of previous death or retirement.

It was proposed that the annuities should be receivable by monthly payments and according to one of four alternative methods arranged to suit varying needs. These are (i) an ordinary annuity for the life of the grantee; (ii) the same, with continuation to his executor until the total annuity payments shall equal the contributions, or rather the purchase price as accumulated up to age 65; (iii) an annuity for the life of the grantee, with continuation of half the amount to his widow, if any; and (iv) the same, with special provision as in method (ii).

It was to be a provision of the Charter of Incorporation that the business should be conducted without profit to the stockholders, and that the Association should issue only non-participating policies; but it is understood that this would not prevent the ultimate division of surplus profits, if any, among insuring members. The Carnegie Foundation proposes to find the share capital and an initial reserve fund, to pay all expenses incurred beyond the free interest income, and to guarantee certain rates of interest on surrendered annuity policies. Among some concluding practical suggestions made by the Joint Commission were (i) that contributions should be made jointly by the institution and the teacher, with liberty to vary the proportion, (ii) that while the deferred annuities should be compulsory, the insurances should be optional, and

(iii) that on migration to an institution within the scheme or likely to be so the benefit of the policies should enure to the member.

These suggestions follow very closely the practice of the Federated Universities System in this country, though in one respect they fall short of it. Under the latter system the whole benefit of past contributions enures to the member upon withdrawal arising from any cause, whereas the Carnegie Scheme apparently contemplates that some option may be reserved to the employing institutions in respect of its own past contributions in the event of withdrawal from teaching altogether. I will postpone what I have to say on the general principle here involved, which appears to me to lie at the root of all pension problems and to have received hitherto quite insufficient attention.

In view of the difficulty of dealing with disablement pensions in an insurance scheme, as has been found in the working of the Federated Universities System and similar schemes in this country, the following conclusions of the Joint Commission on this matter may be quoted.

> The basis of these pensions or the definition of disability may vary all the way from "total and permanent" disability, as understood by insurance companies that have a disability clause in their policies, to a much more liberal definition under which a man is regarded as disabled when he is physically or mentally disqualified to continue the profession of teaching. The Commission thinks that in devising a permanent pension system the first thing to be aimed at is sufficiency of funds; and the definition of disability should be made with due regard to the funds available for the purpose of paying disability pensions. It is the sense of the Commission that as liberal definition as seems to be consistent with the funds available for this purpose is desirable.

The risk of disablement, the Joint Commission add, will not be dealt with by the proposed new Insurance Association, but will be provided for by the Carnegie Foundation. I have not found in the report any definite statement that this proposal has been accepted by the Trustees, but it seems clear that the idea is adumbrated with their consent and is under consideration.

Viewing the financial position of the Carnegie Foundation as it exists to-day, apparently a complete actuarial valuation has not been made, but it is estimated that its outgoings on

the basis of present commitments for the next 45 years will be £13,800,000. The similar estimate of its prospective income is £6,800,000. There is thus, on a 45 years' budget, a prospective deficit of £7,000,000 as the result of granting or promising pensions during the last 12 years. It is thus not surprising that the Trustees of the Foundation should have acted on the conclusions of the Joint Commission and, as mentioned in the extract already quoted, decided (i) to promote, for the use of all members of the profession to be appointed in the future and of all institutions not yet associated with the Foundation, a new pension scheme based on contributions in advance by employers and employed; and (ii) to concentrate their own financial efforts mainly on the problem of their existing commitments. In this task they are exceptionally fortunate in being able to fall back upon another great foundation, with ample funds, the Carnegie Corporation of New York. The latter has agreed to furnish the Carnegie Foundation for Teaching with (i) for the capital and initial fund of the new Insurance Association, £200,000; (ii) for existing teachers still outside, £200,000; and (iii) for creation of Reserve towards its own accruing liabilities a present sum of £2,200,000, and an annual contribution of £120,000. These munificent contributions may not, of course, supply the whole deficiency, and they are subject to the condition that the Trustees will revise their rules and, by contributions from the younger men, or in some other way, ensure a future actuarial balance. In calculating its position over the next 45 years, the Foundation appears to have reckoned as an asset only the value of its future assured income. The *corpus* of its funds, now nearly £3,000,000, is apparently to be preserved intact if possible, so that it will still occupy a strong financial position. There thus appears to be a fair prospect that it will be able, as its own heavy but limited liabilities are gradually reduced, to assume some responsibility for providing disablement pensions in accordance with the principles laid down by the Joint Commission, namely, that the contingency of disablement is best provided for apart from superannuation, and that the definition of disablement may properly vary to some extent with the funds available.

(*To be continued.*)

On a method of Approximate Valuation. By Alfred Henry, F.I.A.

In times like the present, when the saving of clerical labour is a matter of some importance, it may be of interest to consider a method of approximate valuation which gives results very close to the true values and at the same time shortens the mechanical work very materially.

Briefly, the method aims at substituting for the numerous multiplications involved in the process of valuation, simple summations of the data which can be readily and rapidly performed on an adding machine.

If the valuation factor (*i.e.*, single premium, annuity, &c.) be denoted by $F(x)$ and u_x be the amount to be valued at age x, then $\Sigma u_x F(x)$ gives the total liabilities or assets as the case may be.

If now $F(x)$ can be expressed as $a+bx+cx^2$, then the above total becomes $a\Sigma u_x+b\Sigma xu_x+c\Sigma x^2u_x$. Clearly in this form, the valuation can be expressed in terms of the continuous summation of the u_x column from the bottom upwards, for:

$$\sum_{r=0}^{r=n}{}^2 u_r = \sum_{r=0}^{r=n} u_r(r+1)$$

and

$$\sum_{r=0}^{r=n}{}^3 u_r = \sum_{r=0}^{r=n} u_r \frac{(r+1)(r+2)}{2}$$

Therefore

$$\sum_{r=0}^{r=n} r^2u_r = 2\sum_{r=0}^{r=n}{}^3 u_r - 3\sum_{r=0}^{r=n}{}^2 u_r + \sum_{r=0}^{r=n} u_r$$

and

$$\sum_{r=0}^{r=n} ru_r = \sum_{r=0}^{r=n}{}^2 u_r - \sum_{r=0}^{r=n} u_r$$

Accordingly the valuation formula can be expressed as

$$2c\Sigma^3u_r + (b-3c)\Sigma^2u_r + (a-b+c)\Sigma u_r$$

It follows that, given the values of the constants a, b and c for any particular table, the valuation can be made from successive summations of the u_r column.

In the examples that follow, the values of the constants have been found for old-established offices by weighting the equations with the numbers at each fifth age as obtained from

the appropriate model office. Actually the values can, of course, be found from any three ages, as three equations only are necessary to find three unknowns. It was, however, thought better to use weighted equations for all ages, afterwards combining these equations in three groups in order to give the three equations necessary for the finding of the required values.

This method gives the best results, because the final result is accurate *either* if the distribution is the same as that of the model office *or* if the fit is perfect. The resulting deviations are the product of the misfit and the deviation from the normal, and this product must tend to be small. Thus if u' be the actual numbers and u be the normal numbers, and F, F′ be the true and the approximate values of the function, so that $u' = u + \delta u$ and $F' = F + \delta F$

we want $(u + \delta u)F$

and we get $(u + \delta u)(F + \delta F)$

The error is $(u + \delta u)\delta F = \delta u \,.\, \delta F$ since $u\delta F$ is made zero in the fitting.

In order to test the accuracy of this assumption, a curve was fitted to the H^M 3 per-cent annuity table, weighting the values by the numbers living according to Mr. King's model office of 50 years of age. The values so obtained were employed to make a valuation of the model office of 10 years of age, with the following result:

True Valuation £656,445, Approximate Valuation £655,356. The error, on this severe test, is £1,089, or one-sixth of one per-cent.

The following table is of interest, as showing the closeness of the approximation at individual ages:

Values of a_x.

Age x	True Value	Approximate Value	Age x	True Value	Approximate Value
22	21·656	22·370	57	11·353	11·401
27	20·582	20·899	62	9·498	9·707
32	19·373	19·395	67	7·712	7·981
37	18·037	17·860	72	5·975	6·223
42	16·566	16·293	77	4·512	4·433
47	14·923	14·694	82	3·290	2·612
52	13·188	13·064	87	2·393	·758

Except at the extreme ages (where the errors are in opposite directions) the difference between the two values is nowhere more than ·27. As has already been pointed out, these errors tend to balance and, in fact, the method has been found to give very good results, even in cases where the form of the valuation function did not lend itself readily to be fitted by a parabolic curve.

For endowment assurances valued by the Z method, the mean valuation age corresponding to each unexpired term has been taken from Dr. Buchanan's model office. Although the adoption of such ages introduces a further source of error, it is interesting to note that, in practice, a balance of error is secured.

In the following examples the data have been taken at random from the Board of Trade returns, with the figures cut down so as to conceal their identity. Comparison is made in each case with the valuation results as published. The data are shown in the appendix; these are summed continuously twice from the bottom upwards, and a complete cast of the final column gives the value of the third summation. In each case the test is a severe one, as re-assurances are not deducted.

Example 1.—Office A. Whole life. With profits. Valuation of sums assured and bonuses. O^M 3 per-cent.

The origin is taken at age 15 and therefore any policies existing at younger ages must be treated as age 15 and summations made up to and including that age.

$$a = \cdot 242378 \qquad b = \cdot 00880808 \qquad c = \cdot 0000127587$$

$$\Sigma = 18584 \qquad \Sigma^2 = 678541 \qquad \Sigma^3 = 14280130$$

$$2c\Sigma^3 = 364\cdot 3$$
$$(b-3c)\Sigma^2 = 5950\cdot 7$$
$$(a-b+c)\Sigma = 4340\cdot 9$$

Value of sum assured and bonuses ... ... $10655\cdot 9$ True value $= 10646$

Example 2.—Office A. Whole life. With profits. Valuation of tabular premiums. O^M 3 per-cent.

The limited payment premiums must, of course, be valued separately.

Premiums payable throughout life.

$a=25{\cdot}511892 \qquad b=-{\cdot}302413 \qquad c=-{\cdot}000438052$

$\Sigma=462{\cdot}8 \qquad \Sigma^2=17797{\cdot}7 \qquad \Sigma^3=388074{\cdot}4$

$$\begin{aligned} 2c\Sigma^3 &= -\ \ 340{\cdot}0 \\ (b-3c)\Sigma^2 &= -\ 5358{\cdot}9 \\ (a-b+c)\Sigma &= \ \ 11946{\cdot}7 \end{aligned}$$

Value of premiums ... ... 6247·8

Limited payment premiums.

The origin was taken at unexpired term 0; the value of a is therefore zero. As the summations are taken, not to the origin, but to the next higher term, the formula is slightly different.*

$a=0 \qquad b={\cdot}8918174 \qquad c=-{\cdot}01007234$

$\Sigma=66{\cdot}6 \qquad \Sigma^2=725{\cdot}6 \qquad \Sigma^3=5551{\cdot}0$

$$\begin{aligned} 2c\Sigma^3 &= -\ 111{\cdot}8 \\ (b-c)\Sigma^2 &= \ \ 654{\cdot}4 \\ a\Sigma &= \ \ \ \ 0{\cdot}0 \end{aligned}$$

Value of limited payment premiums 542·6

Value of "whole life" premiums (as above) 6247·8

Total value of premiums... ... 6790·4 True value=6802

Example 3.—Office B. Endowment assurances with profits. Valuation of sums assured and bonuses. O^M 3 per-cent.

The policies being grouped according to year of maturity, the origin is taken as the valuation year. The first group of policies is, therefore, at unit distance from the origin, and the formula to be used is the same as that used in valuing the limited payment premiums.

$a={\cdot}99227 \qquad b=-{\cdot}0243313 \qquad c={\cdot}0002376118$

$\Sigma=23141 \qquad \Sigma^2=446969 \qquad \Sigma^3=5327002$

$$\begin{aligned} 2c\Sigma^3 &= \ \ 2531{\cdot}5 \\ (b-c)\Sigma^2 &= -10981{\cdot}5 \\ a\Sigma &= \ \ 22962{\cdot}1 \end{aligned}$$

Value of sum assured and bonuses ... ... 14512·1 True value=14499

* Alternatively if 0 be inserted as the value at the origin and the summations be taken up to and including the zero value, the original formula can be used without modification.

Example 4.—Office B. Endowment assurances with profits. Valuation of tabular premiums. O^{M} 3 per-cent.

$$a = -\cdot2295125 \qquad b = \cdot8351065 \qquad c = -\cdot00815726$$
$$\Sigma = 926\cdot7 \qquad \Sigma^2 = 16451\cdot4 \qquad \Sigma^3 = 184579\cdot3$$

$$2c\Sigma^3 = -\ 3011\cdot3$$
$$(b-c)\Sigma^2 = 13872\cdot9$$
$$a\Sigma = -\ 212\cdot7$$

Value of tabular premiums $10648\cdot9$ True value$=10677$

Example 5.—Office C. Valuation of annuities on female lives. $O^{[af]}$ 3 per-cent.

The distribution of annuity business is such that little weight is given to ages under 40; the fit of the approximate curve is therefore not so good for these ages, and it is best to treat all annuitants under age 40 as being of that age.

The origin is taken at age 40 and summations are therefore taken up to and including that age.

$$a = 19\cdot544538 \qquad b = -\cdot3703151 \qquad c = \cdot00065768$$
$$\Sigma = 13108 \qquad \Sigma^2 = 370693 \qquad \Sigma^3 = 6303147$$

$$2c\Sigma^3 = 8290\cdot9$$
$$(b-3c)\Sigma^2 = -138004\cdot6$$
$$(a-b+c)\Sigma = 261052\cdot5$$

Value of annuities ... ... $131338\cdot8$ True value$=130946$

N.B.—The difference is almost entirely due to the one large contract at age 78.

Example 6.—Friendly Society D. Value of benefits.

$$a = 32\cdot14096 \qquad b = \cdot259632 \qquad c = -\cdot01020274$$
$$\Sigma = 1000 \qquad \Sigma^2 = 20100 \qquad \Sigma^3 = 297367$$

$$2c\Sigma^3 = -\ 6068$$
$$(b-3c)\Sigma^2 = 5834$$
$$(a-b+c)\Sigma = 31871$$

Value of benefits ... ... 31637 True value$=31636$

Example 7.—Friendly Society E. Value of benefits.

$$a = 32{\cdot}14096 \qquad b = {\cdot}259632 \qquad c = -{\cdot}01020274$$

$$\Sigma = 100 \qquad \Sigma^2 = 2{\cdot}704 \qquad \Sigma^3 = 44597$$

$$\begin{aligned} 2c\Sigma^3 &= -\ 910 \\ (b-3c)\Sigma^2 &= \ \ 785 \\ (a-b+c)\Sigma &= 3187 \end{aligned}$$

Value of benefits ... ... 3062 True value = 3062

The above examples could be multiplied indefinitely, but they are sufficient to show that the method yields results which are sufficiently accurate for practical purposes. In particular, for the valuation of such items as sums assured, bonuses, gross premiums or annuities payable, where the data do not vary with the basis of valuation, the results are much superior to those yielded by the use of a model office. If the constants are available for several mortality tables at different rates of interest, then a single set of summations of the data will enable valuations on several different bases to be made in a few minutes. For a net premium valuation, some approximation to the net premiums on the new basis would be necessary.

It should be added that the method is not entirely empirical, as it would be easy to calculate from the approximate annuity values, rates of mortality which would satisfy exactly the assumptions as to the shape of the annuity curve.

Office A.

Age	Sums Assured and Bonuses	Tabular Premiums	Age	Sums Assured and Bonuses	Tabular Premiums
15	1	...	56	608	16·4
16	5	·1	57	493	13·5
17	9	·1	58	399	9·6
18	4	·1	59	397	11·0
19	3	·1	60	320	9·5
20	7	·1	61	437	13·0
21	4	·1	62	345	10·9
22	23	·3	63	359	12·7
23	63	1·1	64	222	5·9
24	65	1·1	65	281	8·1
25	85	1·4	66	339	11·6
26	34	·6	67	228	7·2
27	56	1·1	68	732	29·1
28	196	3·6	69	175	5·7
29	253	4·8	70	155	4·1
30	250	4·7	71	183	8·8
31	194	3·6	72	201	4·5
32	295	5·9	73	111	1·9
33	177	3·5	74	83	2·1
34	389	8·4	75	68	2·1
35	390	7·5	76	55	1·6
36	306	6·4	77	79	1·3
37	356	7·2	78	52	1·7
38	518	12·1	79	40	·7
39	452	9·2	80	44	·5
40	445	11·1	81	31	·5
41	584	11·5	82	30	·5
42	378	8·2	83	46	1·1
43	382	7·7	84	8	·3
44	539	12·3	85	20	·1
45	495	9·4	86	6	·1
46	465	10·5	87	18	·3
47	427	8·6	88	6	...
48	535	13·2	89	7	·2
49	476	10·1	90	2	·1
50	510	14·8	91	10	·1
51	436	11·2	92	4	·1
52	751	17·2	93	1	...
53	478	11·7	94	...	...
54	478	13·7	95	...	...
55	466	11·3	96	9	·3

Premiums payable for limited periods.

Unexpired Term	Tabular Premiums	Unexpired Term	Tabular Premiums	Unexpired Term	Tabular Premiums
1	2·4	12	2·4	23	·5
2	1·8	13	3·6	24	·6
3	2·0	14	1·5	25	·6
4	2·2	15	4·5	26	·2
5	5·6	16	1·2	27	·1
6	4·4	17	5·4	28	·4
7	4·2	18	1·9	29	·3
8	6·9	19	3·2	30	...
9	2·4	20	·5	31	·2
10	1·8	21	·3	...	...
11	5·1	22	·4	...	...

Office B.

n*	Sums Assured and Bonuses	Tabular Premiums	n*	Sums Assured and Bonuses	Tabular Premiums
1	131	7·0	24	879	30·2
2	138	7·8	25	997	34·1
3	266	13·8	26	743	23·8
4	296	16·1	27	705	22·1
5	260	15·5	28	661	20·9
6	434	23·5	29	689	21·1
7	351	17·4	30	470	14·0
8	463	22·6	31	399	11·6
9	535	24·6	32	356	9·9
10	627	30·8	33	293	8·1
11	727	32·9	34	227	6·1
12	892	40·4	35	265	7·0
13	742	33·5	36	138	3·5
14	806	37·2	37	115	2·9
15	941	43·4	38	102	2·5
16	1,044	47·9	39	70	1·7
17	1,027	44·4	40	75	1·8
18	1,052	44·9	41	19	·4
19	1,189	49·9	42	29	·7
20	1,209	48·9	43	18	·4
21	1,019	39·3	44	4	·1
22	897	32·7	45	...	...
23	833	29·1	46	8	·2

* n = Year of maturity − Year of valuation. Valuation date is 31 December.

Office C.

Age	Amount of Annuity	Age	Amount of Annuity	Age	Amount of Annuity
21	17	44	540	67	410
22	...	45	27	68	275
23	...	46	71	69	127
24	10	47	50	70	203
25	...	48	116	71	126
26	...	49	235	72	615
27	...	50	226	73	402
28	...	51	150	74	526
29	...	52	36	75	78
30	...	53	61	76	251
31	...	54	281	77	237
32	...	55	249	78	2,817
33	...	56	401	79	29
34	11	57	290	80	313
35	64	58	357	81	151
36	...	59	247	82	...
37	25	60	46	83	145
38	...	61	637	84	312
39	...	62	145	85	89
40	73	63	410	86	115
41	...	64	413	87	...
42	...	65	439	88	24
43	...	66	186	89	50

Societies D *and* E.

Age last Birthday	Number of Members		Age last Birthday	Number of Members	
	Society D	Society E		Society D	Society E
16	35	...	43	30	3
17	28	2	44	23	6
18	33	1	45	20	4
19	37	1	46	12	...
20	20	...	47	14	8
21	14	3	48	19	3
22	22	...	49	13	1
23	38	...	50	19	4
24	33	1	51	13	2
25	36	2	52	7	...
26	28	...	53	10	5
27	23	2	54	12	3
28	30	...	55	12	...
29	19	...	56	8	...
30	28	4	57	10	1
31	32	3	58	7	3
32	17	...	59	14	2
33	15	1	60	10	...
34	24	4	61	7	...
35	27	2	62	3	2
36	22	6	63	...	1
37	31	4	64	8	...
38	24	3	65	5	1
39	20	3	66	6	...
40	28	1	67	...	1
41	26	5	68	3	...
42	21	1	69	4	1

Decreasing Debts on Endowment Assurances. By Charles H. Ashley, A.I.A., *Actuary of the London and Manchester Assurance Company.*

In his useful note on this subject (*J.I.A.*, vol. xlix, p. 266), Mr. Todhunter starts off with the assumption that "in any "case, the extra premium that would be charged has been "determined according to the usual office practice," and so proceeds to find the debt. But I suggest, with Mr. Coutts (p. 278), that it is no more unscientific, having regard to the paucity of data on which to calculate extra risks, to assess a sub-standard life "by a debt in the first instance and measure the extra risk in this form."

The decreasing debt system has limitations in practice which I consider should be defined at the outset. Tables

could then be constructed with due regard thereto, showing what mortality curves can be provided for and what extra premiums will correspond.

Accordingly, in the hope of developing the study of the subject of extra rating (a subject which seems to have grown in importance during the war, I submit the appended tables based on the following practical limitations, namely:

(1) That it is not advisable in practice to offer acceptance of a sub-standard risk subject to a debt which, during the first few years, is so large as to deprive the assured of the fundamental feature of life assurance, namely, protection against premature death. If the defect is such that the initial debt would have to be so large as say £75 per-cent, it is better simply to quote the extra premium.

(2) That at no time during the currency of the policy shall the nominal sum assured, less the current debt, be less than the total amount of the premiums already paid.

(3) That the annual reduction of the debt be deferred until after the first 5, 7 or 10 years: this with the object of keeping down the magnitude of the initial debt.

(4) That the annual reduction after the period of deferment shall be at least the amount of the renewal premium paid. This follows, I believe, an American system.

The headings in the tables indicate the endowment periods and debts, and the columns show:

(1) The Extra Pure Premiums, calculated at $3\frac{1}{2}$ per-cent interest, on the assumption that the extra mortality is so distributed as to produce policy-values equalling normal O^{M} 3 per-cent policy-values at the end of each year, bonus excluded. (If the extra premium, which is payable for the whole endowment period, is paid in lieu of the debt, 10 per-cent might be added for expenses). Formula adopted

$$q_{[x]+n-1}(1-{}_{n}V_{x\overline{t|}})=q'_{[x]+n-1}(1-n\text{th debt }-{}_{n}V_{x\overline{t|}})$$

the select values of q and q' being assumed to be in the same ratio as the O^{M} values.

(2) The number of Years Rating-up which will produce, on the basis of $O^{[M]}$ Mortality, $3\frac{1}{2}$ per-cent interest, premiums for the increasing assurance approximately equal to premiums for a level £100 at true ages.

(3) n, where $100(P_{[x+n]\overline{t|}} - P_{[x]\overline{t|}})$ by $O^{[M]}$ $3\frac{1}{2}$ per-cent, equals the tabulated extra pure premiums.

(4) The Constant Additions to $O^{[M]}$ Rates of Mortality which will produce premiums for the increasing assurance approximately equal to the normal $O^{[M]}$ premiums for £100. Interest $3\frac{1}{2}$ per-cent.

In the tables various types of extra mortality which are covered by the specified debts are indicated. Four diagrams are appended to illustrate the mortality curves involved in respect of age at entry 35.

The *thick* curve is the standard for comparison, $O^{[M]}$.

The *circles* curve represents rates of mortality which produce, under the debt system, claims each year approximately equal to the $O^{[M]}$ claims under normal policies.

The *thin* curve represents Hardy's method of assuming sub-standard mortality such as to give normal reserves, and is seen to be applicable to sub-standard lives which may be expected to show increasing extra mortality during the first few years, thereafter dropping to the normal. Though I cannot think that any class of sub-standard risks would show such a pronounced maximum mortality as this "thin" curve, I gather from enquiries amongst medical officers that curves with that tendency (more probably the "circles" curve) are quite likely, and a recent history of typhoid fever or of abscess in the abdomen are suggested as examples. Mr. Burn presumes such a type on p. 207 of the 5th Congress Transactions, and the Associated Scottish Life Offices' Investigation found that among publicans the extra risk increases with duration of assurance to a maximum and then diminishes. It must not be forgotten that among standard lives there are many who develop progressive diseases, and the reduction of sub-standard lives through earlier death must surely leave the residue nearer to the standard.

The *dotted* curve represents the too-popular method of assessing a life by stating that it is equivalent to one a certain number of years older. The way such a curve soars after a few years suggests to me its rarity in practical experience. Nevertheless, impaired lives are accepted which undoubtedly

annum.

Age at Entry	Initial Debt £52 10s.				In	—				—				Age at Entry
	Extra pure prem. on described basis (1)	Years rating up (2)	n (3)	Constant addition to $q^{O[M]}$ covered by the debt (4)	Extra pure prem. on described basis (1)	Extra pure prem. on described basis (1)	Years rating up (2)	n (3)	Constant addition to $q^{O[M]}$ covered by the debt (4)	Extra pure prem. on described basis (1)	Years rating up (2)	n (3)	Constant addition to $q^{O[M]}$ covered by the debt (4)	
	s. d.				s. d									
20	8 11	...	...	...	5 7	...	...	...	...	...	...	...	...	20
25	9 7	20	23	·007	6 0	...	...	...	...	...	...	...	...	25
30	10 9	...	...	...	6 8	...	...	...	...	...	...	...	...	30
35	12 4	15	17	·009	7 8	...	...	...	...	...	...	...	...	35
40	14 11	...	...	...	9 3	...	...	...	...	...	...	...	...	40
45	19 2	12	13	·013	11 10	...	...	...	...	...	...	...	...	45
50	25 10	...	...	...	16 0	...	...	...	...	...	...	...	...	50
55	36 5	10	11	·028	22 8	...	...	...	...	...	...	...	...	55

annum.

Age at Entry	Initial Debt £49 10s.				In	—				—				Age at Entry
	s. d.				s. d									
20	8 9	...	...	.	6 3	...	...	...	...	...	...	...	...	20
25	9 7	18	20	·007	6 10	...	...	...	...	...	...	...	...	25
30	10 10	...	...	...	7 8	...	...	...	...	...	...	...	...	30
35	12 8	13	15	·009	9 0	...	...	...	...	...	...	...	...	35
40	15 8	...		...	11 2	...	...	...	...	...	...	...	...	40
45	20 6	10	12	·015	14 7	...	...	...	...	...	...	...	...	45
50	28 3	...	..	...	20 1	...	...	...	...	...	...	...	...	50

er annum.

Age at Entry	Initial Debt £49 10s.				In	Initial Debt £22 10s.				—				Age at Entry
	s. d.				s. d	d.								
20	11 2	...	...	...	8 5	1	...	...	...	...	...	...	...	20
25	12 4	18	20	·008	9 3	4	7	7	·002	...	...	...	...	25
30	14 0	...	...	...	10 7	7	...	...	...	...	...	...	...	30
35	16 10	14	15	·011	12 9	2	4	5	·002	...	...	...	...	35
40	21 6	...	...	...	16 2	0	..	...	...	...	...	...	...	40
45	28 9	12	13	·019	21 8	3	3	4	·004	...	...	...	...	45

er annum.

Age at Entry	Initial Debt £49				Initi	Initial Debt £28				Initial Debt £24 10s.				Age at Entry
	s. d.				s. d	d.				s. d.				
20	9 8	...	...	...	7 11	10	...	...	...	2 4	...	...	...	20
25	10 9	15	17	·007	8 10	2	7	8	·002	2 6	7	7	·002	25
30	12 5	...	...	...	10 2	7	...	..	...	2 10	...	...	...	30
35	15 2	12	13	·010	12 5	5	4	5	·003	3 6	4	4	·002	35
40	19 8	...	...	...	16 1	7	..	...	...	4 5	...	...	...	40

15-year Endowment Assurance. Initial Debts fixed during first 5 years, then reducing by £7 10s. per annum.

Age at Entry	Initial Debt £52 10s.				Initial Debt £45				Initial Debt £37 10s.				Initial Debt £30				Initial Debt £22 10s.				—				—				—				Age at Entry
	Extra pure prem. on described basis	Years rating up	n	Constant addition to $q^{O[M]}$ covered by the debt	Extra pure prem. on described basis	Years rating up	n	Constant addition to $q^{O[M]}$ covered by the debt	Extra pure prem. on described basis	Years rating up	n	Constant addition to $q^{O[M]}$ covered by the debt	Extra pure prem. on described basis	Years rating up	n	Constant addition to $q^{O[M]}$ covered by the debt	Extra pure prem. on described basis	Years rating up	n	Constant addition to $q^{O[M]}$ covered by the debt	Extra pure prem. on described basis	Years rating up	n	Constant addition to $q^{O[M]}$ covered by the debt	Extra pure prem. on described basis	Years rating up	n	Constant addition to $q^{O[M]}$ covered by the debt	Extra pure prem. on described basis	Years rating up	n	Constant addition to $q^{O[M]}$ covered by the debt	
	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	
	s. d.				s. d.				s. d.				s. d.				s. d.																
20	8 11	...	...	...	5 7	...	...	...	3 6	...	...	...	2 2	...	...	...	1 3	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	20
25	9 7	20	23	·007	6 0	16	19	·005	3 10	12	15	·003	2 5	9	11	·002	1 5	6	8	·001	...	...	...	...	...	...	...	...	...	...	...	...	25
30	10 9	...	...	...	6 8	...	...	...	4 3	...	...	...	2 8	...	...	...	1 7	..	...	...	...	...	...	...	...	...	...	...	...	...	...	...	30
35	12 4	15	17	·009	7 8	11	13	·006	4 10	8	10	·004	3 0	6	7	·003	1 10	4	5	·002	...	...	...	...	...	...	...	...	...	...	...	...	35
40	14 11	...	...	...	9 3	...	...	...	5 10	...	...	...	3 7	...	...	...	2 2	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	40
45	19 2	12	13	·013	11 10	8	10	·009	7 6	6	7	·006	4 8	4	5	·004	2 10	3	3	·002	...	...	...	...	...	...	...	...	...	...	...	...	45
50	25 10	...	...	...	16 0	...	...	...	10 2	...	...	...	6 5	...	...	...	3 10	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	50
55	36 5	10	11	·028	22 8	7	8	·018	14 4	5	6	·011	9 0	4	4	·007	5 4	2	3	·005	...	...	...	...	...	...	...	...	...	...	...	...	55

20-year Endowment Assurance. Initial Debts fixed during first 7 years, then reducing by £5 10s. per annum.

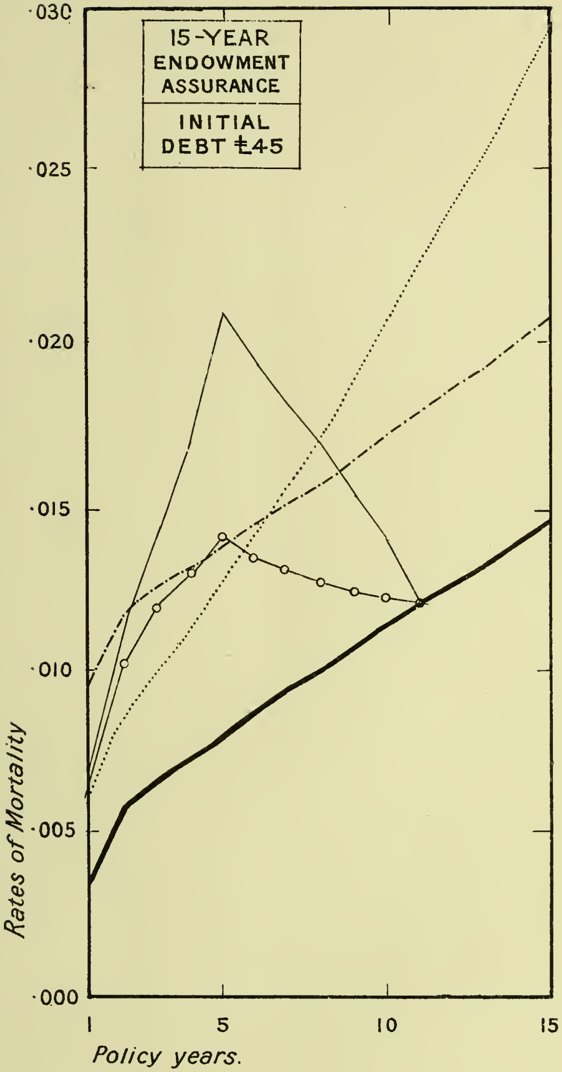
	Initial Debt £49 10s.				Initial Debt £44				Initial Debt £38 10s.				Initial Debt £33				Initial Debt £27 10s.				Initial Debt £22				—				—				
	s. d.				s. d.				s. d.				s. d.				s. d.				s. d.												
20	8 9	...	...	.	6 3	...	...	..	4 6	...	...	...	3 3	...	...	...	2 4	...	...	...	1 7	...	...	...	...	...	...	...	...	...	...	...	20
25	9 7	18	20	·007	6 10	15	17	·005	4 11	13	14	·004	3 7	11	12	·003	2 7	9	10	·002	1 9	7	7	·001	...	...	...	...	...	...	...	...	25
30	10 10	...	...	...	7 8	...	...	...	5 7	...	...	...	4 0	...	...	...	2 10	...	...	...	1 11	...	...	...	...	...	...	...	...	...	...	...	30
35	12 8	13	15	·009	9 0	11	12	·006	6 6	9	10	·005	4 8	7	8	·004	3 4	5	6	·003	2 3	4	4	·002	...	...	...	...	...	...	...	...	35
40	15 8	...		...	11 2	...	...	...	8 0	...	...	...	5 9	...	...	...	4 1	...	...	...	2 9	...	...	...	...	...	...	...	...	...	...	...	40
45	20 6	10	12	·015	14 7	8	9	·011	10 6	6	7	·008	7 6	5	6	·006	5 3	4	4	·004	3 7	3	3	·003	...	...	...	...	...	...	...	...	45
50	28 3	...	..	...	20 1	...	...	...	14 5	...	...	...	10 3	...	...	...	7 3	...	...	...	4 10	...	...	...	...	...	...	...	...	...	...	...	50

25-year Endowment Assurance. Initial Debts fixed during first 10 years, then reducing by £4 10s. per annum.

	Initial Debt £49 10s.				Initial Debt £45				Initial Debt £40 10s.				Initial Debt £36				Initial Debt £31 10s.				Initial Debt £27				Initial Debt £22 10s.				—				
	s. d.				s. d.				s. d.				s. d.				s. d.				s. d.				s. d.								
20	11 2	...	...	...	8 5	...	...	...	6 5	...	...	...	4 11	...	...	...	3 9	...	...	...	2 10	...	...	...	2 1	...	...	...	...	...	...	...	20
25	12 4	18	20	·008	9 3	16	18	·007	7 1	14	15	·005	5 5	12	13	·004	4 2	10	11	·003	3 2	8	9	·002	2 4	7	7	·002	...	...	...	...	25
30	14 0	...	...	...	10 7	...	..	...	8 1	...	...	...	6 2	...	...	...	4 8	...	...	...	3 6	...	...	...	2 7	...	...	...	...	...	...	...	30
35	16 10	14	15	·011	12 9	12	13	·009	9 8	10	11	·007	7 5	8	9	·005	5 8	7	8	·004	4 3	5	6	·003	3 2	4	5	·002	...	...	...	...	35
40	21 6	...	...	...	16 2	...	...	...	12 4	...	...	...	9 5	...	...	..	7 3	...	...	...	5 5	...	...	...	4 0	..	...	...	...	...	...	...	40
45	28 9	12	13	·019	21 8	10	11	·015	16 6	8	9	·012	12 7	6	7	·009	9 6	5	6	·007	7 2	4	5	·005	5 3	3	4	·004	...	...	...	...	45

30-year Endowment Assurance. Initial Debts fixed during first 10 years, then reducing by £3 10s. per annum.

	Initial Debt £49				Initial Debt £45 10s.				Initial Debt £42				Initial Debt £38 10s.				Initial Debt £35				Initial Debt £31 10s.				Initial Debt £28				Initial Debt £24 10s.				
	s. d.				s. d.				s. d.				s. d.				s. d.				s. d.				s. d.				s. d.				
20	9 8	...	...	...	7 11	...	...	...	6 6	...	...	...	5 4	...	...	...	4 5	...	...	...	3 7	...	...	...	2 10	...	...	...	2 4	...	...	...	20
25	10 9	15	17	·007	8 10	13	15	·006	7 3	12	13	·005	5 11	10	12	·004	4 10	9	11	·003	3 10	8	9	·003	3 2	7	8	·002	2 6	7	7	·002	25
30	12 5	...	...	...	10 2	...	...	...	8 4	...	...	...	6 10	..	...	...	5 7	...	...	...	4 6	...	..	...	3 7	...	..	...	2 10	...	...	...	30
35	15 2	12	13	·010	12 5	10	11	·009	10 2	9	10	·007	8 4	7	8	·006	6 9	6	7	·005	5 6	5	6	·004	4 5	4	5	·003	3 6	4	4	·002	35
40	19 8	...	...	...	16 1	...	...	...	13 2	...	...	...	10 8	...	...	...	8 8	...	...	...	7 0	...	...	...	5 7	..	...	...	4 5	...	...	...	40



15-YEAR ENDOWMENT ASSURANCE
INITIAL DEBT £45
·030
·025
·020
·015
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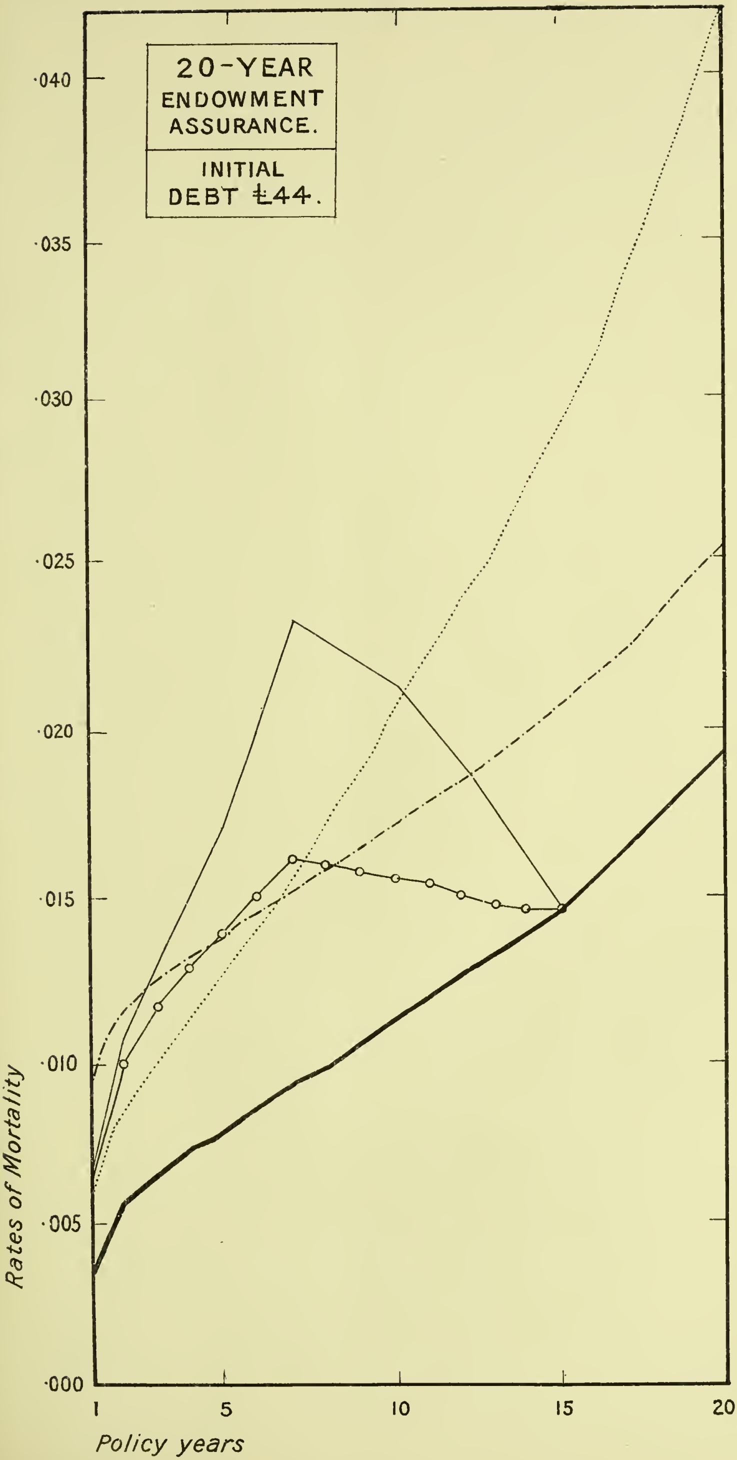
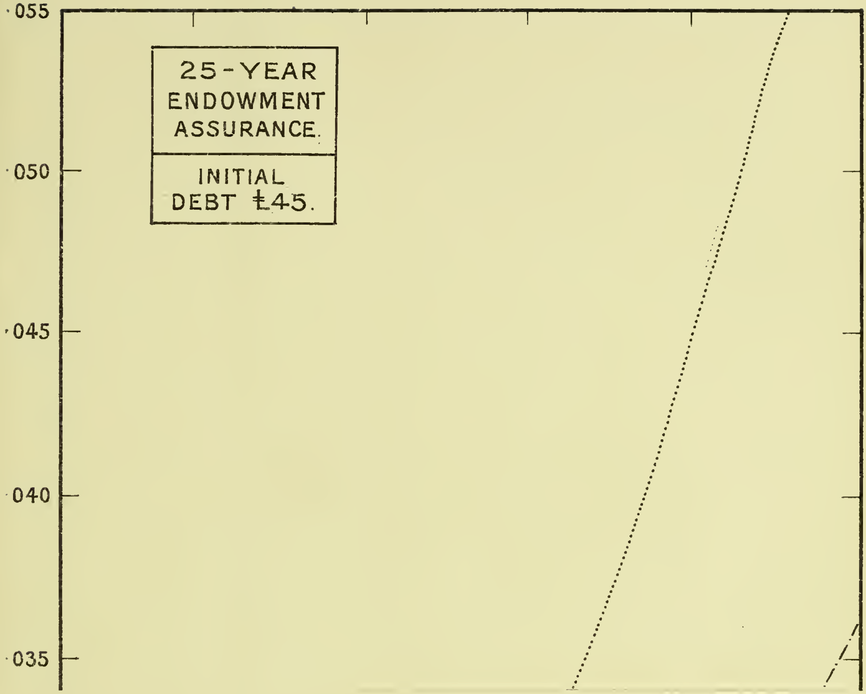
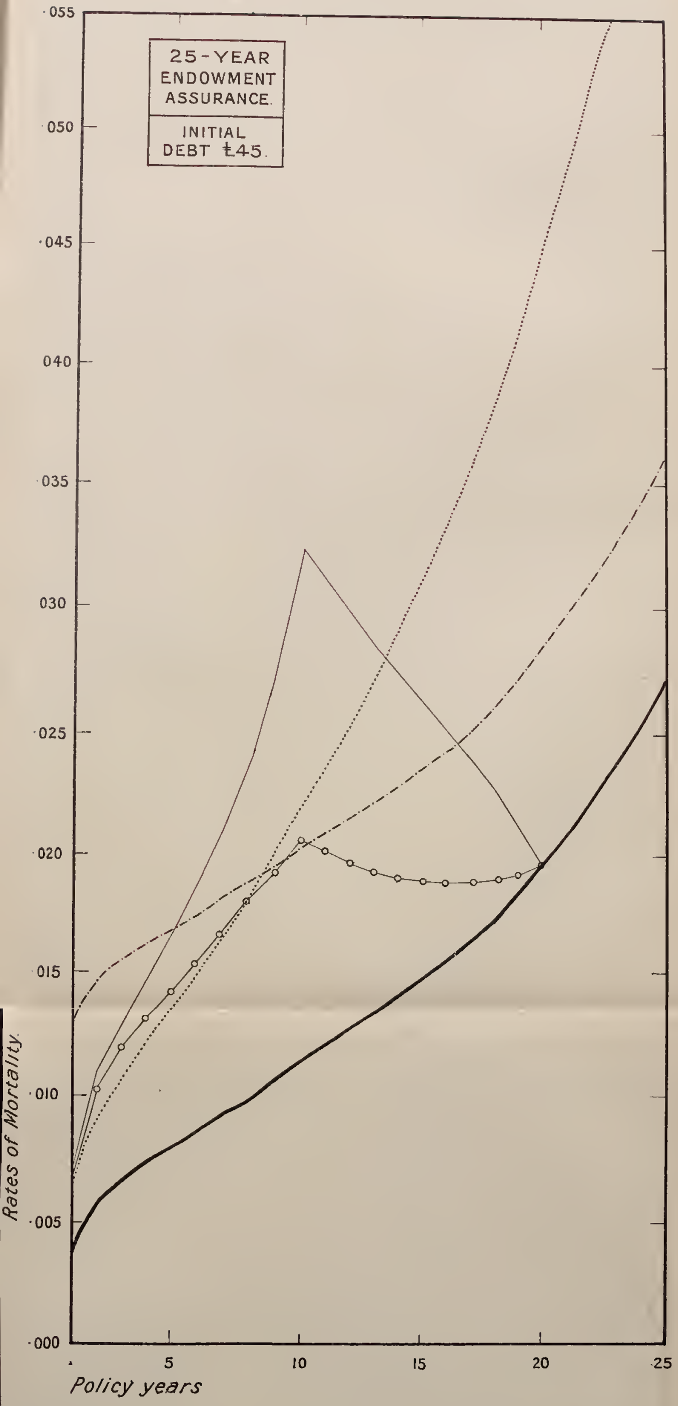
·005
·000
1
5
10
15
Rates of Mortality
Policy years.


20-YEAR
ENDOWMENT
ASSURANCE.
INITIAL
DEBT £44.
·040
·035
·030
·025
·020
·015
·010
·005
·000
1
5
10
15
20
Rates of Mortality
Policy years



·055
·050
·045
·040
·035
25-YEAR ENDOWMENT ASSURANCE.
INITIAL DEBT £45.



25-YEAR ENDOWMENT ASSURANCE.
INITIAL DEBT £45.
·055
·050
·045
040
·035
030
·025
·020
015
·010
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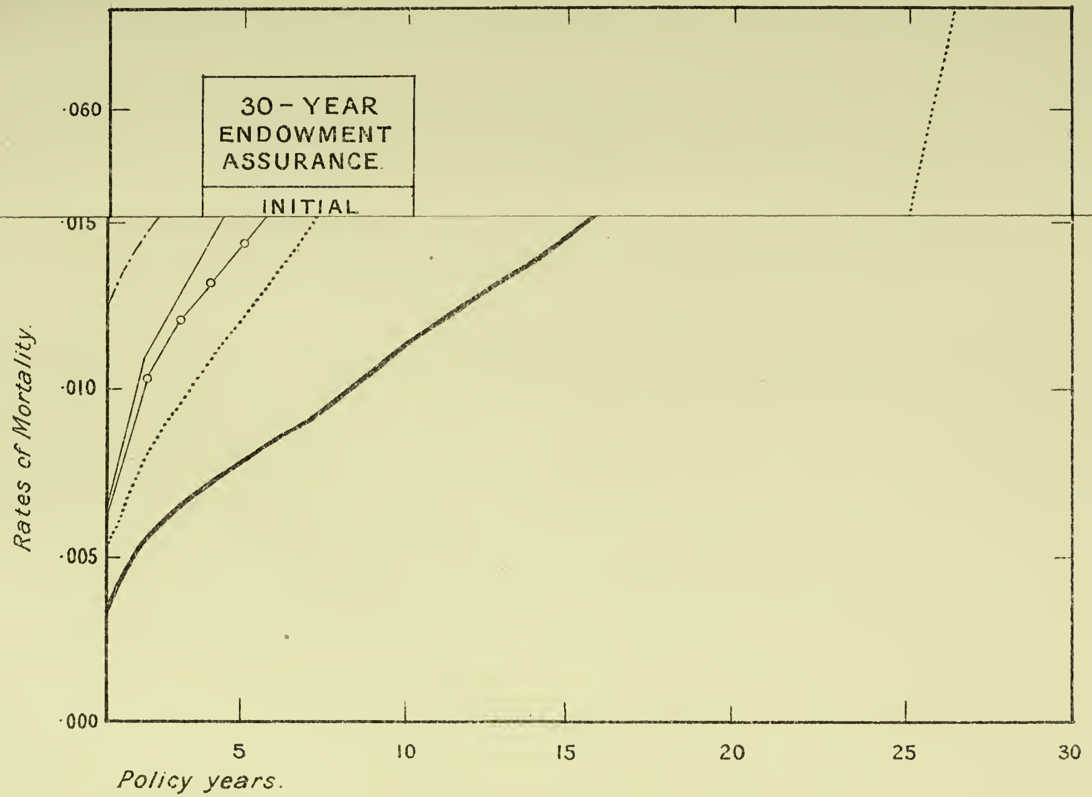
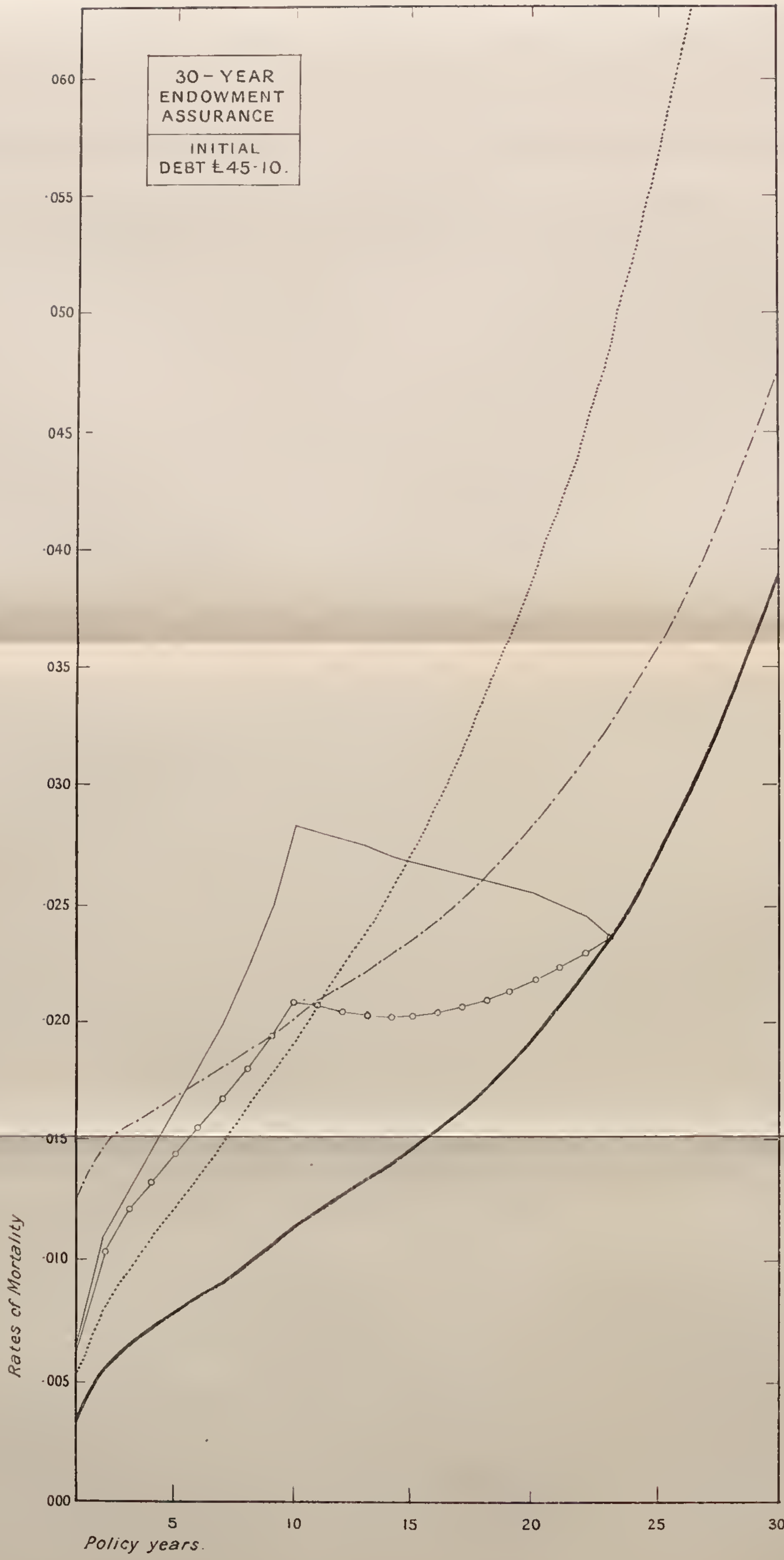
·000
5
10
15
20
25
Rates of Mortality.
Policy years


30 - YEAR
ENDOWMENT
ASSURANCE.
INITIAL.
·060
·015
·010
·005
·000
Rates of Mortality.
5
10
15
20
25
30
Policy years.



30-YEAR ENDOWMENT ASSURANCE
INITIAL DEBT £45·10.
·060
·055
·050
·045
·040
·035
·030
·025
·020
·015
·010
·005
·000
Rates of Mortality
5
10
15
20
25
30
Policy years.

show an increasing extra mortality, though less steeply than that represented by a substantial addition to the age.

The *broken* curve is perpendicularly equidistant from the standard and represents a constant addition to the normal rate of mortality such as will justify the specified debt. Probably a large number of lives will be put into this class as a result of injuries received in the war.

The debt system can, of course, be used for any particular incidence of extra risk because the consideration is received at the outset. In other words, the assured "pays" in the early years for the extra risk, although that extra risk may not mainly arise till the later years. On the other hand, the system is limited in its usefulness to the lighter extra risks. No additional labour need be incurred at the periodical valuations, since all types of extra mortality will be operating and we shall get a good average result by valuing all the endowment assurances as normal for the full sums assured.

Industrial offices get large numbers of endowment assurance proposals on the lives of unmarried young men to whom the investment element appeals more strongly than the immediate death benefit. To those who are found to be sub-standard lives, a decreasing debt ought to be more acceptable than the payment of an extra premium. Indeed, in such cases, it seems to me that we can avoid any "moral hazard" by not quoting an extra premium. In other cases the assured should always be given the option of paying the corresponding extra premium.

Mathematical Symbols.

STUDENTS of Lewis Carroll's "Pillow Problems" may recall the use in that sleep-dispelling work—and the interesting explanation in the preface to the Fourth Edition—of two geometrical symbols for the sine and cosine. These particular symbols have never, so far as we know, been used by other mathematicians. But they are illustrative of an evolutionary process that has been going on from the earliest times—or at any rate since the introduction or systematization of algebraical symbolism by Diophantus. With the continual development of mathematical analysis, the necessity arises for new symbols, not only to save the frequent repetition of verbal expressions,

but also to convey the idea of an operation or relation to the mind more rapidly than is possible by the use of words. Of the symbols tentatively introduced to meet this need many—such as the above-mentioned symbols for the sine and cosine—do not survive their birth; others—as in the case of the symbol for equality, first introduced by Recorde* in 1557, but not generally adopted until more than a century later—became established only after a long struggle for existence; others again attain a limited acceptance in some special branch of research; a few are almost immediately accepted as supplying a long-felt want.† This process of evolution in mathematical symbolism has been very active during the last 20 or 30 years, and symbols which are unfamiliar to students brought up on the older books are now commonly employed, without explanation or definition, in modern text-books and tracts even of an elementary nature. A symbol of which the meaning is not accurately understood is as serious an obstacle to study as if the work in which it is used were written in an unknown language, and it is hoped, therefore, that the appended list of a few modern symbols may assist actuarial students to pursue their reading both of current actuarial work and of analytical research bearing thereon.

The principle governing the use of mathematical symbols is clear. A "recognized" symbol may be employed without explanation; any other—whether one that although frequently used cannot be held to have been generally accepted by mathematicians, or a special symbol employed for a special

* "to avoide the tediouse repetition of these woordes : is equalle to : I will "sette as I doe often in woorke use, a paire of paralleles, or Gemowe" [*i.e.*, twin] "lines of one lengthe, thus : ═══ : bicause noe 2 thynges can be moare equalle."—The Whetstone of Witte, 1557.

† For some information as to the origin of the more common symbols in algebra reference may be made to W. W. Rouse Ball's *Short Account of the History of Mathematics*, and to the article on *Algebra* in the Encyclopædia Britannica.

We are indebted to Mr. G. J. Lidstone for the following interesting extract from Arbogast's *Du Calcul des Dérivations*:

"Il a fallu introduire des signes nouveaux; j'ai donné une attention particulière à cet objet, persuadé que le secret de la puissance de l'Analyse consiste de la choix et l'emploi heureux des signes simples et caractéristiques de la chose qu'ils doivent representer. Je me suis prescrit à cet egard les règles suivantes: (1) de rendre les notations le plus qu'il était possible analogues à des notations reçues; (2) de ne point introduire des notations inutiles et que j'aurais pu remplacer sans confusion par des notations déjà en usage; (3) de les choisir très-simples, en y faisant entrer cependant toutes les variétés qu'exigent les différences des operations."

purpose—should be defined. The difficulty consists, in the absence of any final authority, in determining what constitutes "recognition", and it is increased by the fact that in certain instances two symbols seem to have attained recognition, or at any rate are employed by different writers without explanation, for the same purpose. Our list deals only with symbols (of comparatively recent introduction) which are certainly recognized or may be considered to be on the border line. It is not suggested that the list is complete, and some of the symbols have been in use among mathematicians for many years but have been included because they may be unfamiliar to many actuarial students.

A consideration of some practical importance is that it is desirable, when possible, to avoid the use of symbols that entail the necessity for the process known to printers as "justification", and thus to reduce the expense both of composition and of corrections. Several of the symbols in our list—(1) and (4) for example—are instances of alternative symbols that have been devised with this object, and such expressions as "arctan", "sinh", "cosh", &c., belong to the same category. It may be remembered that a good many years ago the British Association appointed a committee of eminent mathematicians to report "on Mathematical Notation " and Printing, with the view of leading mathematicians to " prefer, in optional cases, such forms as are more easily put " into type, and of promoting uniformity of notation." The resulting report (see *J.I.A.*, vol. xx, p. 355) would not appear to have had so much effect as might be desired on the practice of mathematical authors, and this may perhaps have been due to the fact that the committee, although expressing a pious opinion that uniformity of notation would "tend towards a common language in mathematics", confined their attention almost entirely to the first part of their reference and recommended some forms of notation in preference to others "from the printing, and not from the scientific point of view." It may be suggested that the scientific point of view should take priority, that recommendations made from that point of view by a strong committee of leading mathematicians might do much to promote uniformity, and that while some forms of notation must always necessitate justification, printers would probably be able to a very considerable extent to meet the requirements of mathematicians, if those

requirements were based on uniformity of practice, by stocking certain symbols that have at present to be specially composed.

SOME MODERN MATHEMATICAL SYMBOLS AND THEIR MEANINGS.

/ ... ... Used instead of the horizontal line to represent a fraction. Thus $a/b=\frac{a}{b}$. This is not generally used when the fraction itself is the subject of transformation. Thus we should write

$$e^x=1+x+x^2/2\,!+x^3/3\,!+\ \ldots$$

but less commonly

$$(x-a)/(x^3-a^3)=1/(x^2+ax+a^2).$$

! ... ... Used instead of $\underline{|}$ to represent a factorial. Thus $n\,!\ =\underline{|n}=n(n-1)(n-2)\ldots 1$, where n is supposed positive and integral.

$\equiv$ ... ... Denotes an *identity* which is true for all values of the variables as distinguished from an *equation* which is true only when some of the variables have particular values; thus $(a+b)^2\equiv a^2+2ab+b^2$, *i.e.*, "is transformable into" $a^2+2ab+b^2$ "by applying the laws of algebra, without "any assumption regarding the operands "involved" (Chrystal Algebra, Part I, 5th Edition, p. 22). Sometimes used in a more restricted sense to indicate merely that one (usually shorter) expression *represents* another; thus

$$v_q\equiv a_0+a_1q+a_2q^2+\ \ldots\ .\ +a_nq^n;$$

$$x^3\equiv x\times x\times x.$$

$\sim$ ... ... Formerly placed between two quantities to indicate that the smaller (algebraically) is to be subtracted from the larger; thus $a\sim b$ denotes $a-b$ or $b-a$ according as a or b is the larger. But now practically

superseded in this sense by the symbol | | (*see below*), and used in the theory of infinite series to denote a certain kind of approximate equality called "asymptotic."

D ... ... Used to denote differentiation. Thus $Df(x) \equiv df(x)/dx$ or $f'(x)$. Note that in Boole's *Finite Differences* (1st Edition) and in *Text-Book*, Part II, this symbol is used to denote the operator $(1+\Delta)$ or $e^{d/dx}$, now more generally represented by E.

$\binom{m}{r}$ ... ... Used instead of ${}_rC_m$, (m, r) or $\frac{m!}{r!(m-r)!}$. Thus, for example,

$$(1+x)^m = 1 + mx + \binom{m}{2}x^2 + \binom{m}{3}x^3 + \ldots\ldots$$

exp. ( ) ... Used to represent e raised to the power whose index is the quantity in the brackets. Thus

$$\text{exp. } (a+bx+cx^2+\ldots\ldots) = e^{a+bx+cx^2+\ldots\ldots}$$

| | ... ... The modulus of; thus $|z|$ is the modulus of z. If z is a complex quantity equal to $x+yi$ [where i is the imaginary quantity $\sqrt{-1}$], the modulus of z is the positive value of $\sqrt{x^2+y^2}$. Thus $|x+yi| = +\sqrt{x^2+y^2}$. Hence when $y=0$ and z is wholly real we have $|x| = +\sqrt{x^2}$, *i.e.*, the modulus of a real quantity is its absolute value irrespective of sign, or its numerical value taken positively; the symbol is consequently used to denote the absolute value, without reference to sign, of a magnitude which may be either positive or negative. For many purposes the modulus of a complex quantity takes the place of the numerical value of a real quantity, especially in questions relating to convergency. *Note.*—Care must be taken not to confuse the symbol for the *modulus* with a similar symbol sometimes used as

an abbreviated notation for a *determinant*. For example, the determinant of the third order whose successive rows are a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3, where the constituents of the principal diagonal are a_1, b_2, c_3, is represented in this notation by $|a_1b_2c_3|$, which in another connection would mean the modulus of the product $a_1b_2c_3$.

$=\!\!\!\backslash\!\!=$ ... ... Is not equal to; has any other value than. (In continental works the symbol $\lessgtr$ is often used with this meaning.) *Example.*

$$\int x^m dx = x^{m+1}/(m+1) \text{ if } m \neq -1.$$

If $m=-1$, the function $x^{m+1}/(m+1)$ becomes infinite, but the indefinite integral $=\log_e x$.

$\ngtr$, or $\leqslant$ or $\leqq$ Is not greater than; is either equal to or less than.

$\nless$, or $\geqslant$ or $\geqq$ Is not less than; is either equal to or greater than.

Note.—Where an actual inequality is in question the form $\ngtr$ or $\nless$ is usually the more convenient,* but the later forms are much used to denote the limits of a variable. Thus to express that x may take any value over the range -1 to $+1$ except $+1$ itself, we should write $-1 \leqslant x < +1$. For example, if this be the case the series $x - x^2/2 + x^3/3$ . . . . *ad inf.* converges to the sum $\log_e(1+x)$.

$\stackrel{\frown}{=}$ or $\risingdotseq$ ... Is approximately equal to. Thus $\delta \risingdotseq i+d$. Such a symbol is much required in actuarial work. Both forms are used without special definition in the *Phil. Mag.* The latter form, proposed by Lord Kelvin, is more easily distinguished from the symbol $=$, and is more easily written.

* Rouse Ball, however, remarks (*loc. cit.* p. 242) that the symbols $\neq$, $\ngtr$, $\nless$, "are, I believe, now rarely used outside Great Britain; they were employed, if "not invented, by Euler. The symbols $\geqq$ and $\leqq$ were introduced by P. Bouguer "in 1734."

$\longrightarrow$ ... ... Tends to the value; has the limiting value.

Examples.

As $x \longrightarrow \infty$, $\left(1+\frac{n}{x}\right)^x \longrightarrow e^x$, and $e^{-x} \longrightarrow 0$.

As $h \longrightarrow 0$, $(u_{x+h}-u_x)/h \longrightarrow \frac{d}{dx}u_x$.

O( ) ... ... Is of the order of; ultimately preserves a finite ratio to (*i.e.*, as the argument or independent variable $\longrightarrow \infty$). Thus

$$\frac{15n+19}{n^3+1} \text{ is } O\left(\frac{1}{n^2}\right).$$

The strict analytical definition is as follows. If u_x and v_x are two functions of x such that there is a number, say n, such that the modulus $|u_x : v_x|$ is $<K$ whenever $x > n$, where K is finite and independent of n, we say that u_x is $O(v_x)$.

o( ) ... ... Ultimately vanishes in comparison with (*i.e.*, as the argument or independent variable $\longrightarrow \infty$). Thus

$$\frac{\log x}{x} \longrightarrow 0 \text{ as } x \longrightarrow \infty; \text{ i.e., } \log x \text{ is } o(x).$$

LEGAL NOTES.

By WILLIAM CHARLES SHARMAN, F.I.A., *Barrister-at-Law.*

Relief from Super-tax in respect of life premium paid under Contract.

UNDER Sec. 36 (1) of the Finance Act, 1916, it is enacted that relief from super-tax shall not be given in respect of premiums payable under life assurances and deferred annuities.

An interesting point in this connection arose in the case of *Earl Howe* v. *Commissioners of Inland Revenue*, which will be found reported in *The Times* of 10 July 1918, but had not appeared in the official reports at the time when these Notes were written.

The case was an appeal from a decision of the Commissioners for the Special Purposes of the Income Tax Acts in respect

of a claim by Lord Howe to deduct amounts paid by him as premiums on life assurance policies from his assessment to super tax.

The facts are as follows:

For the purpose of raising certain sums of money, Lord Howe granted his interest in certain estates and also assigned a certain policy of life assurance on his own life to an assurance company by way of mortgage. He covenanted to pay (1) interest on the sums advanced by him; (2) the premiums on the life policy. He also gave a right to the assurance company to pay the premiums if he himself neglected to do so, and to charge them on the mortgaged property. It was contended on behalf of Lord Howe that there was no distinction in principle between the annual premium payable and the mortgage interest. The Commissioners decided that Lord Howe was not entitled to deduct the amounts of such premiums from his assessment.

In giving judgment in favour of the appellant, Mr. Justice Sankey said: "If the case came before me unembarrassed by "authority, I am bound to say that a careful consideration of "the sections of the Income Tax Acts would lead me to the "conclusion that the contention of the Crown is correct. The "matter has, however, recently come before the Irish Courts in "the case of *Commissioners of Inland Revenue* v. *Lord Massy*, "and two out of three judges decided that the contention as "advanced by Lord Howe in the present case is correct. An "appeal was brought before the Irish Court of Appeal, but "unfortunately, before judgment was given, Lord Massy died, "and his executors did not continue the case; I think, therefore, "in these circumstances I ought to follow the majority judgment "in the Irish Courts, and, as a result, the appeal succeeds."

I understand an appeal has been lodged against the decision and the case will shortly come before the Court of Appeal.

When is an endowment policy a policy of life assurance?

The question whether certain forms of endowment policies could be correctly termed policies of life assurance, again came before the Court in the case of *In re National Standard Life Assurance Corporation*, 118 L.T.R. 621.

The decision in this case is not inconsistent with previous decisions in regard to somewhat similar kinds of assurance reported in these Notes under the names of *Flood* v. *Irish*

Provident Assurance Company, Limited (*J.I.A.*, vol. xlvii, p. 298), and *Joseph* v. *Law Integrity Insurance Company, Limited* (*J.I.A.*, vol. xlvii, p. 300).

The National Standard Life Assurance Corporation was incorporated in 1906, and £20,000 was deposited in the Court of Chancery in respect of the life assurance business of the company.

On 6 July 1916, an order to wind up the company compulsorily was made, and the liquidator took out a summons to have it determined whether certain assurances were "policies on human life" within the meaning of the Assurance Companies Act, 1909, so as to entitle the holders thereof to participate in the statutory deposit.

The assurances were divided into four classes, as follows: In consideration of certain premiums, the Corporation bound itself

(1) to pay a fixed sum to the assured at a fixed date if he should be then living, with provision for the payment of a smaller amount to the legal personal representatives in the case of death;

(2) to pay a fixed sum at a fixed date without any reference to the death of the assured. In this case the premiums were payable during the whole period, and it was open to the legal personal representatives to continue to pay the premiums;

(3) to pay a sum at a fixed date, such sum varying according to whether certain options were exercised, the legal personal representatives having the right to surrender the policy within six months of the death of the holder, and to receive the total amount of premiums paid;

(4) to pay a fixed sum at a fixed date with a provision that should the holder die before the date of maturity, the legal personal representatives should be entitled to a return of the premiums paid.

In the course of his judgment, Neville, J., said: "In my opinion "there is only one class of policy mentioned in the summons "which raises any special difficulty, and that is the policy "falling under the third class, whereby the company assures "payment of a sum of money at a fixed date, and gives an "option to the legal personal representatives to determine the "contract on the death of the assured, on the exercise of which "option all premiums are to be repaid. The policies falling "within the fourth class clearly come within the statutory "definitions, because in that case the money is payable upon

"the death of the assured unless something else intervenes in "the shape of a new contract between the legal personal "representatives of the assured and the company. Unless "that happens, the premiums are to be returned on the "death of the assured. In the case of policies falling under "the third class, however, the premiums are not to be returned "to the legal personal representatives unless they choose to "determine the contract, which they have power to do; but "in my opinion, the contention was right that the determination "of the contract by them is a 'contingency dependent on human "life', and falls within the statutory definition contained in "Sect. 30 (*a*) of the Assurance Companies Act, 1909. The "result, therefore, is that all the policies, with the exception of "those falling under the second class, are in my opinion 'policies "on human life', and entitle the holders to rank against the "statutory deposit."

Legal decisions on this point were formerly of some importance owing to their bearing on the question as to what constituted a life insurance policy for the purpose of obtaining relief from income tax. The restrictions contained in recent Finance Acts have, however, considerably modified the importance of this question. At the same time it is interesting to note the distinction drawn between a pure endowment policy with return of premiums in the event of death, and a capital redemption or sinking fund policy.

Annuity payable free of deductions does not avoid deduction of Income Tax.

If it be desired to grant an annuity payable free of income tax, it is necessary that words clearly indicating this intention should be used, and the statement that the annuity is to be paid free of any deduction is not sufficient to avoid deduction of income tax. In the case of *In re Loveless*, 34 T.L.R. 356, the Court of Appeal held that a direction in a will to pay a clear annuity is not to be construed as giving the annuity free of tax. The facts are as follows: The testator, Thomas Henry Loveless, by his will dated 18 July 1902, after giving various legacies and annuities —to be paid free of estate, settlement estate, legacy or succession duty—directed that his trustees should stand possessed of two equal third parts of the residuary trust fund upon trust out of the income thereof to pay to his wife, Edith Jane Loveless, "a clear annuity of £2,000 during her widowhood commencing

"from my death", and in the event of her remarriage then for the remainder of her life "a clear reduced annuity of £1,000 "for her separate use without power of anticipation", such annuities to be paid by equal quarterly payments.

The will contained a power to appropriate out of the fund a sufficient part to answer the annuity payable to his wife, with power to resort to the capital in case of a deficiency. The testator died on 12 November 1916. The trustees took out an originating summons to determine, amongst other questions, whether, according to the true construction of the will, the widow was entitled to receive the annuity free from income tax, or whether she had to bear the income tax on it.

Mr. Justice Eve held, on the authorities, that the word "clear" was properly applicable to the instalments of legacy duty payable in respect of the annuity, and that the widow was not entitled to receive the annuity free of income tax.

The Court of Appeal dismissed the appeal of the widow against this decision.

In giving judgment Lord Justice Swinfen Eady said that the learned judge below had held that the direction to pay a clear annuity did not give an annuity free from income tax, and in his (his Lordship's) opinion he was right in so holding. The Income Tax Act, 1842, provided (Section 102) that on annuities or other annual payments there should be charged yearly for every 20*s.* of the amount thereof the tax payable without deduction, and a proviso contained a power enabling the person authorized to make the annual payment to deduct the tax payable, and the person entitled to receive the annual payment must allow the deduction, as if the amount deducted had actually been paid to him. The Income Tax Act, 1853, Section 40, extended that power, providing that "the person liable to such payment shall be acquitted "and discharged of so much money as the deduction shall "amount unto, as if the amount thereof had actually been "paid unto the person to whom such payment shall have been "due and payable." The Customs and Inland Revenue Act, 1888, Section 24, Sub-section 3, contained a positive direction that the person paying the annuity should deduct the income tax payable thereon.

In his opinion a clear annuity was paid to an annuitant where there was paid, first the income tax payable thereon, and then the balance payable direct to the annuitant, the

two sums making together the full sum payable to the annuitant. The tax was paid out of the annuity, and it was as much paid to the annuitant as if the whole annuity had been paid to her direct without deduction, in which case she would have to pay the income tax thereout herself. In all cases where there was a direction in a will to pay an annuity clear of or free from all deductions, the will ought not by reason of those words to be construed as giving the annuity free of income tax.

Lord Justice Bankes and Mr. Justice Neville concurred.

Ambiguous answers to questions in proposal.

The case of *Yorke* v. *Yorkshire Insurance Company, Limited* (1918), 1 K.B. 662, raised some interesting points with regard to the questions contained in a proposal for life assurance.

The facts are as follows: The insurance company granted a policy of insurance for £1,000, dated 6 January 1917, to one Robert Smith, payable in the event of death occurring before 4 December 1917. The policy was assigned to the plaintiff on 28 February 1917. Robert Smith died on 25 March 1917, and the insurance company refused payment on the ground that certain answers in the proposal were untrue, and that the assured had failed to disclose that he suffered from heart trouble and insomnia, and was addicted to the veronal habit.

The proposal form contained (*inter alia*) the following questions: What illnesses have you suffered? Answer: None of consequence. Do you ordinarily enjoy good health? Answer: Yes. Are you now and have you always been of sober and temperate habits? Answer: Yes.

In giving judgment in favour of the insurance company, Mr. Justice McCardie held that the question as to what illnesses the assured had suffered was not ambiguous and that the answer thereto was not a mere expression of opinion and that the untruth of the answer rendered the policy void. The words "sober and temperate" in a proposal for life insurance must be held, however, to refer only to the use and abuse of alcohol, and not to drug habits. Unless the question be expressly asked, insurance companies must, with respect to the use of drugs by a proposer, rely on the rule of law which requires the disclosure of all material facts known to the proposer which might lead the insurer to refuse the risk.

The United States Moratorium in respect of the Life Assurance Premiums of Officers, Men and Nurses on Active Service.

[On pp. 54–56 of the current volume we gave the more material sections of the Act of the United States Legislature providing for the grant by the Treasury War Risk Insurance Bureau of new convertible term assurances to officers, men and nurses on active service. Our attention has since been drawn by Mr. J. Douglas Watson to the Soldiers' and Sailors' Civil Rights Act, signed by President Wilson on March 8, 1918, by which provision is made for the non-lapsing of existing assurances effected before 1 September 1917 by persons now in military or naval service. *The Commercial and Financial Chronicle* of New York comments as follows on the assurance provisions of the Act: "The bill of the special session" (a Moratorium Bill which was not passed by Congress) "was especially objectionable in the portion relating to life insurance, because it proposed to stop lapsing on policies which had been in force a year and in practice would have undertaken to confiscate the reserve on some policies for the benefit of holders of other policies . . . . When the insurance experts pointed this out to the sub-committee of the Senate, whose members had proposed to rush ahead and forbid lapsing and had not stopped to consider how that might work, they were able to see the point and candid enough to confess it. Then came a proposition that the Government guarantee the companies against loss and protect itself, in turn, by taking a lien on the policies. This proposition, with its crudeness removed, has been followed in the Act as passed. . . . The Article" (*i.e.*, the Insurance Article of the Act) "has far smaller application than was proposed at the special session, and its terms are far different from the broad bar upon lapsing first written into the bill. Any loss occurring will now fall upon the country, as it should; the companies will be subjected to considerable trouble and some expense, but they will be protected, and the scheme is apparently workable. In respect to justice it is far better than the" (British) "Emergency Powers Act which was hastily enacted by Parliament soon after the war began . . . ." The following extracts from the Act will sufficiently indicate the nature of the scheme.—Eds. *J.I.A.*] :

Sec. 401. That the benefits of this Article shall apply to any person in military service* who is the holder of a policy of life insurance when such holder shall apply for such benefits on a form prepared in accordance with regulations which shall be prescribed by the Secretary of the Treasury. . . . The original of such application shall be sent by the insured to the insurer, and a copy thereof to the Bureau of War Risk Insurance. . . .

Sec. 402. That the benefits of this Act shall be available to any person in military service in respect of contracts of insurance in force under their terms up to but not exceeding a face value of $5,000, irrespective of the number of policies held by such person whether in one or more companies when such contracts were made and a premium was paid thereon before September 1 1917. . . .

Sec. 404. That when one or more applications are made under this Article by any one person in military service in respect of insurance exceeding a total face value of $5,000, whether on one or more policies or in one or more companies, and the insured shall not in his application indicate an order of preference, the Bureau of War Risk Insurance shall reject such policies as have the inferior cash surrender value, so as to reduce the total benefits conferred within the face value of $5,000, and where necessary for this purpose shall direct the insurer to divide any policy into two separate policies. . . .

Sec. 405. That no policy which has not lapsed for the nonpayment of premium before the commencement of the period of military service of the insured, and which has been brought within the benefits of this Article, shall lapse or be forfeited for the nonpayment of premium during the period of such service or during one year after the expiration of such period ; *Provided* that in no case shall this prohibition extend for more than one year after the termination of the war.

Sec. 406. That within the first fifteen days of each calendar month after the date of approval of this Act until the expiration of one year after the termination of the war, every insurance corporation or association to which application has been made as herein provided, for the benefits of this Article, shall render to the Bureau of War Risk Insurance a report, duly verified, setting forth the following facts :

First. The names of the persons who have applied for such benefits, and the face value of the policies in respect of which such benefits have been applied for by such persons, during the preceding calendar month ;

Second. A list as far as practicable of the premiums in respect of policies entitled to the benefits of this Article which remain unpaid on the last day of the preceding calendar month . . . .

Fourth. A computation of the difference between the total amount of defaulted premiums . . . . and the total amount of premiums paid after having been previously reported as in

* Including, broadly, as defined by Sec. 101, all officers, men and nurses on active service in the Army or Navy.

default. . . . The final sum so arrived at shall be denominated the monthly difference.

Sec. 408. That the Secretary of the Treasury shall, within ten days thereafter, deliver each month to the proper officer of each insurer, bonds of the United States to the amount of that multiple of $100 nearest to the monthly difference certified in respect of each insurer. . . .

Sec. 409. That the bonds so delivered shall be held by the respective insurers as security for the payment of the defaulted premiums with interest. To indemnify it against loss the United States shall have a first lien upon any policy receiving the benefits of this Article, subject to any lien existing at the time the policy became subject to this Act. . . .

Sec. 410. That in the event that the military service of any person being the holder of a policy receiving the benefits of this Article shall be terminated by death, the amount of any unpaid premiums, with interest at the rate provided for in the policy for policy loans, shall be deducted from the proceeds of the policy and shall be included in the next monthly report of the insurer as premiums paid.

Sec. 411. That if the insured does not within one year after the termination of his period of military service pay to the insurer all past-due premiums with interest thereon from their several due dates at the rate provided for in the policy for policy loans, the policy shall at the end of the year immediately lapse and become void, and the insurer shall thereupon become liable to pay the cash surrender value thereof if any: *Provided* that if the insured is in the military service at the termination of the war, such lapse shall occur and surrender value be payable at the expiration of one year after the termination of the war.

Sec. 412. That at the expiration of one year after the termination of the war there shall be an account stated between each insurer and the United States, in which the following items shall be credited to the insurer:

(1) The total amount of the monthly differences reported under this Article;

(2) The difference between the total interest received by the insurer upon the bonds held by it as security and the total interest upon such monthly differences at the rate of 5 per-cent per annum;

and in which there shall be credited to the United States the amount of the cash surrender value of each policy lapsed or forfeited as provided in Section 411, but not in any case a greater amount on any policy than the total of the unpaid premiums with interest thereon at the rate provided for in the policy for policy loans.

Sec. 413. That the balance in favour of the insurer shall, in each case, be paid to it by the United States upon the surrender by the insurer of the bonds delivered to it from time to time by the Secretary of the Treasury under the provisions of this Article.

Sec. 414. That this Article shall not apply to any policy which

is void or which may at the option of the insurer be voidable, if the insured is in military service, either in this country or abroad, nor to any policy which as a result of being in military service, either in this country or abroad, provides for the payment of any sum less than the face amount thereof or for the payment of an additional amount as premium.

ACTUARIAL NOTE.

A Comprehensive Table for the calculation of yields (gross and net) on Redeemable Securities. By J. R. Armstrong, F.F.A., *of the Scottish Provident Institution.*

THE tables of net yields on redeemable securities after allowing for income tax, as published in *J.I.A.*, vol. li, pp. 28–31, are admirably ingenious and compact. They apply, however, only to bonds bought at a discount, and moreover the double interpolation generally required in their use may involve an accumulation of errors which, while not serious in practice, throws us back on a troublesome calculation in cases where greater exactitude may be desired. The present writer feels that there is room for further treatment of the subject, and, acting on a suggestion from Mr. Lidstone, ventures to submit the appended table, based on the first-difference formula discussed in *J.I.A.*, vol. l, pp. 247–250, by means of which both the gross and the net yield on a bond bought at either a premium or a discount may be calculated very easily and with close accuracy.

The editorial comments on the note just referred to include an alternative formula of Mr. Lidstone's, which the present writer recognizes as a distinct and characteristic improvement on his own. The improvement consists in starting, not from g, the nominal rate, as the writer suggested, but from a *trial* rate $g+x_1$ nearer to the required rate $g+x$ and thus involving interpolation for a smaller interval, with, as a result, a higher degree of accuracy in cases where x is large.

The derivation of the "trial rate" formula from the well known expression for the discount in the price, namely,

$$(g+x-g)a^{g+x}=\mathrm{K}$$

consists in substituting $x_1+(x-x_1)$ for x, where $g+x_1$ is a trial rate, whence we have

$$x=\mathrm{K}\div a^{g+x_1+(x-x_1)}=\mathrm{K}[\mathrm{R}^{g+x_1}+\frac{x-x_1}{h}\Delta\mathrm{R}^{g+x_1}]$$

i.e., $x(1-\frac{\mathrm{K}}{h}\Delta\mathrm{R}^{g+x_1})=\mathrm{K}(\mathrm{R}^{g+x_1}-\frac{x_1}{h}\Delta\mathrm{R}^{g+x_1})$

whence, taking the difference-interval $h=\cdot01$, we have

$$x=\frac{\mathrm{K}(\mathrm{R}^{g+x_1}-100x_1\Delta\mathrm{R}^{g+x_1})}{1-100\mathrm{K}\Delta\mathrm{R}^{g+x_1}} \quad . \quad . \quad . \quad . \quad (1)$$

or adding g so as to obtain the yield direct, we have

$$g+x=\frac{g+\mathrm{K}[\mathrm{R}^{g+x_1}-100(g+x_1)\Delta\mathrm{R}^{g+x_1}]}{1-100\mathrm{K}\Delta\mathrm{R}^{g+x_1}} \quad . \quad . \quad (2)$$

The other essential feature of Mr. Lidstone's suggestion consists in substituting, for the variable trial rate $g+x_1$ a "standard" trial rate I which would probably vary only gradually from time to time and which, if taken at present at say ·05, would give good results so far as gross yields are concerned.

For net yields after allowing for tax, a lower "standard" rate—say 4 per-cent—would be more suitable.

Passing to percentages we thus have, in Mr. Lidstone's notation

$$100i \text{ (gross)}=\frac{100g\pm100\mathrm{K}(\mathrm{R}^{\mathrm{I}}-100\mathrm{I}\Delta\mathrm{R}^{\mathrm{I}})}{1\mp100\mathrm{K}\Delta\mathrm{R}^{\mathrm{I}}}$$

$$=\frac{100g\pm100\mathrm{KA}}{1\mp100\mathrm{KB}} \quad . \quad . \quad . \quad . \quad (3)$$

and $$100i \text{ (net)}=\frac{100g(1-t)\pm100\mathrm{K}(\mathrm{R}^{\mathrm{I}'}-100\mathrm{I}'\Delta\mathrm{R}^{\mathrm{I}'})}{1\mp100\mathrm{K}\Delta\mathrm{R}^{\mathrm{I}'}}$$

$$=\frac{100g(1-t)\pm100\mathrm{KA}'}{1\mp100\mathrm{KB}'} \quad . \quad . \quad . \quad (4)$$

the upper or lower signs being taken according as K is the discount or the premium per unit of the price, and t being the rate of income tax per £.

The functions tabulated in the appended table are 100A and 100B at 5 per-cent and 100A′ and 100B′ at 4 per-cent convertible half-yearly. The process of calculation is obvious; it need only be noted that the functions, being tabulated as

percentages, are to be used with K, the discount or premium per unit of price.

As an example take a 3 per-cent debenture redeemable in 30 years, price 58·49 per-cent. The discount being ·4151 per unit, we have for the gross yield

$$\frac{3+\cdot4151\times2\cdot691}{1-\cdot4151\times\cdot756}=\frac{4\cdot117}{\cdot6862}=5\cdot999 \text{ per-cent,}$$

the true gross yield being 6 per-cent.

For the net yield after allowing for tax at 6*s*. per £, we have, substituting 2·1 for 3 in the numerator and taking A′ and B′ at 4 per-cent

$$\frac{2\cdot1+\cdot4151\times2\cdot885}{1-\cdot4151\times\cdot717}=\frac{3\cdot298}{\cdot7024}=4\cdot696 \text{ per-cent,}$$

the true net yield being 4·694 per-cent.

Had the mean of each of the tabulated functions been used in the calculation of both the gross and the net yield the results would have been 5·987 and 4·690 respectively. The example chosen being a very severe test in view of the unusual disparity between the gross rate and the nominal and trial rates, these last results indicate that for most cases in practice a single set of functions, say at 4½ per-cent, would serve for calculating both the gross and the net yield. The difference between the expressions for the gross and net yields would then be simply $\frac{100gt}{1\mp100\text{K}\Delta\text{R}}$, a very convenient expression for the tax deduction from the gross yield when the latter is ascertainable from prepared tables without requiring to be calculated.

While the quantities A and A′ are tabulated with a view to the direct calculation of the *full* rate, gross or net, the writer may add that personally he prefers the more compact formula (1), giving x, the addition to the nominal rate. In this case the quantities required are simply 100R at a convenient rate, say 4 per-cent (convertible half-yearly), and the values of 100ΔR for successive 1 per-cent intervals up to, say, 6 per-cent. The trial rate can then be easily varied at pleasure.

The general formula

$$\frac{g\pm\text{K}(\text{R}^{i}-100i\Delta\text{R}^{i})}{1\mp100\text{K}\Delta\text{R}^{i}}$$

invites comparison with Mr. Todhunter's well known approximate expression for the yield, namely,

$$\frac{g \pm \frac{K}{n}}{1 \mp K\frac{n+1}{2n}}.$$

On the assumption of constant first differences

$$K(R^i - 100i\Delta R^i) = KR^0 = K/n$$

as in Mr. Todhunter's formula. Similarly $\frac{n+1}{2n}$ in the latter is a first approximation to $100\Delta R$. It will be seen from the values tabulated that the errors of the factors of K in Mr. Todhunter's formula are positive and negative in the numerator and the denominator respectively. Hence the values of the numerator and denominator err in the same direction, with the result that, within limits, the expression gives a fair approximation. Closely approximate expressions for R^i and ΔR^i for periods from 10 to 50 years and for rates of interest from 3 per-cent to 6 per-cent may however be obtained as follows: Taking selected values of R within these limits and deducting from them $\frac{1}{n} + \cdot 0053(100i)$ we are left with a residue the quotient of which by $n(100i)^2$ lies, for the most part, between 6 and 7 in the sixth decimal place. We may therefore write

$$100R^i = 100n^{-1} + \cdot 53(100i) + 2n(100i)^2 \div 3000 \text{ approx.} \quad (5)$$

$$100\Delta R^i = \cdot 53 + 2n(200i+1) \div 3000 \quad \text{,,} \quad (6)$$

$$100(R^i - 100i\Delta R^i) = 100n^{-1} - 2n100i(100i+1) \div 3000 \quad \text{,,} \quad (7)$$

$2 \div 3000 = \cdot 000\dot{6}$ being taken as the most convenient factor nearly midway between ·0006 and ·0007. The substitution of (7) and (6) for $100n^{-1}$ and $\frac{n+1}{2n}$ respectively in Mr. Todhunter's formula would not only extend its range considerably, but make it practically as accurate in most cases as if the actual tabular values of the functions were used, especially with a trial rate.

As an illustration, reverting to the above example and using formula (1), we have, at trial rate 5 per-cent:

$$100R = 3{\cdot}333 + 2{\cdot}65 + {\cdot}5 = 6{\cdot}483$$

$$100\Delta R = {\cdot}53 + {\cdot}22 = {\cdot}75$$

$$(5-3) \times 100\Delta R \qquad = 1{\cdot}5$$

$$4{\cdot}983$$

and $$\frac{{\cdot}4151 \times 4{\cdot}983}{1 - {\cdot}4151 \times {\cdot}75} = 3{\cdot}004 = 100x$$

whence we have as the gross yield $100(g+x) = 6{\cdot}004$ per-cent an error of ·004.

Again, at 4 per-cent,

$$100R = 3{\cdot}333 + 2{\cdot}12 + {\cdot}32 = 5{\cdot}773$$

$$100\Delta R = {\cdot}53 + {\cdot}18 = {\cdot}71$$

$$(4 - 2{\cdot}1) \times 100\Delta R \qquad = 1{\cdot}349$$

$$4{\cdot}424$$

and $$\frac{{\cdot}4151 \times 4{\cdot}424}{1 - {\cdot}4151 \times {\cdot}71} = 2{\cdot}604$$

which added to 2·1, the net nominal rate, gives 4·704 per-cent as the net yield, an error of ·010.

It may be added that n^{-1}, *i.e.*, R^0, is the first term in the expansion of $a_{\overline{n}|}^{-1}$, while $\cdot 53i$ is the mean value of the second term in the expansion for $n = 10$ to 50.

The above expressions are not, of course, put forward with any idea of their being generally used in place of the actual values of the functions in question. But they are so nearly accurate, are so easily remembered, and lend themselves so readily to an occasional rapid calculation independently of tables that the writer thinks it worth while to mention them.

Years	4 per-cent		5 per-cent	
	100A′	100B′	100A	100B
2	49·969	·639	49·954	·642
2½	39·961	·618	39·942	·622
3	33·286	·605	33·262	·610
3½	28·516	·597	28·488	·602
4	24·936	·592	24·904	·598
4½	22·150	·588	22·114	·595
5	19·920	·586	19·880	·594
5½	18·093	·586	18·050	·594
6	16·570	·586	16·522	·595
6½	15·280	·586	15·228	·596
7	14·173	·587	14·118	·598
7½	13·212	·588	13·153	·600
8	12·371	·590	12·308	·602
8½	11·628	·592	11·561	·605
9	10·966	·594	10·895	·608
9½	10·373	·596	10·299	·611
10	9·839	·598	9·761	·614
11	8·914	·603	8·829	·620
12	8·141	·608	8·048	·627
13	7·484	·614	7·384	·634
14	6·919	·620	6·812	·641
15	6·428	·626	6·314	·648
16	5·996	·632	5·875	·656
17	5·613	·638	5·486	·663
18	5·272	·644	5·138	·670
19	4·964	·650	4·825	·678
20	4·687	·656	4·541	·685
21	4·434	·662	4·283	·693
22	4·204	·668	4·047	·700
23	3·992	·675	3·830	·707
24	3·797	·681	3·629	·714
25	3·617	·687	3·444	·721
26	3·450	·693	3·272	·729
27	3·294	·699	3·112	·736
28	3·149	·705	2·962	·742
29	3·013	·711	2·822	·749
30	2·885	·717	2·691	·756
31	2·765	·723	2·568	·763
32	2·653	·729	2·451	·769
33	2·546	·735	2·342	·775
34	2·445	·740	2·238	·782
35	2·350	·746	2·140	·788
36	2·260	·751	2·047	·794
37	2·174	·757	1·959	·800
38	2·092	·762	1·875	·806
39	2·014	·768	1·795	·812
40	1·940	·773	1·720	·817
41	1·869	·778	1·647	·823
42	1·801	·783	1·578	·828
43	1·737	·789	1·513	·833
44	1·675	·794	1·450	·838
45	1·616	·798	1·390	·843
46	1·559	·803	1·333	·848
47	1·504	·808	1·278	·853
48	1·452	·813	1·226	·858
49	1·402	·817	1·176	·862
50	1·353	·822	1·128	·867

CORRESPONDENCE.

ORIGINAL TABLES.

To the Editors of the Journal of the Institute of Actuaries.

Dear Sirs,—Having on several occasions had to estimate the value of functions by the $O^{[M]}$ Table at 5 per-cent interest, I have prepared a table of annuities on this basis, and, as it may be of some service to other members of the profession, I send you the table referred to herewith.

In constructing this table the ultimate value of D_x was made up by a continuous process from the relation

$$\log D_{x+1} = \log D_x + \log vp_x$$

the value of $\log p_x$ being taken from the table published in the official volume to five places. Throughout the whole construction, therefore, five place logarithms only were used.

The value of $D_{[x]}$ was then constructed—again by five-place logarithms—directly from the value of $(\log v^x + \log l_{[x]})$ and thereafter the tables of both $D_{[x]}$ and $D_{[x]+10}$ were checked at decennial ages by the arithmometer.

From the relation $\log D_{[x]+n+1} = \log D_{[x]+n} + \log vp_{[x]+n}$ complete tables of $D_{[x]+n}$ were made up—the values of $\log D_{[x]+10}$ so found agreeing with the values of log D ultimate first obtained.

Having by summation constructed tables of $N_{[x]}$ and N and thence constructed—in duplicate by the arithmometer—tables of $a_{[x]}$, $a_{[x]:\overline{60}|}$, $a_{[x]:\overline{40}|}$ and $a_{[x]:\overline{20}|}$ to serve as check values: the tables were then completed by the continuous process

$$a_{[x]:\overline{n}|} = a_{[x]:\overline{n-1}|} + D_{[x]+n} \, . \, D_{[x]}^{-1}$$

In using the values of $\log vp$ the necessary corrections were made to counteract the ignoring of the sixth and seventh places of decimals in the value of $\log v$: but as $\log p$ was, as has been said, taken only to five places of decimals, the resultant annuity-values may differ by one in the third place from values derived from $\operatorname{colog} p$ as tabulated to seven places in the "Account of Principles and Methods."

I am,

Yours faithfully,

C. KEITH GRANGER.

30, *Renfield Street, Glasgow.*
16 *July* 1918.

O[M] $a_{[x]\overline{n}|}$ 5 per-cent.

Duration	10	11	12	13	14	15	16	17	18	19	Duration
	16·916	16·863	16·808	16·750	16·689	16·626	16·559	16·490	16·417	16·341	
0	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000	0
1	·950	·950	·950	·950	·950	·950	·950	·950	·950	·950	1
2	1·851	1·851	1·851	1·851	1·851	1·851	1·851	1·851	1·851	1·851	2
3	2·705	2·705	2·705	2·705	2·705	2·705	2·705	2·705	2·705	2·704	3
4	3·515	3·515	3·515	3·514	3·514	3·514	3·514	3·513	3·513	3·513	4
5	4·282	4·281	4·281	4·281	4·281	4·280	4·280	4·280	4·279	4·279	5
6	5·008	5·008	5·007	5·007	5·007	5·006	5·005	5·005	5·004	5·004	6
7	5·696	5·695	5·695	5·694	5·694	5·693	5·692	5·692	5·691	5·690	7
8	6·347	6·346	6·346	6·345	6·344	6·343	6·342	6·341	6·340	6·339	8
9	6·963	6·962	6·961	6·961	6·960	6·958	6·957	6·956	6·955	6·953	9
10	7·546	7·545	7·544	7·543	7·542	7·540	7·539	7·537	7·537	7·534	10
1	8·098	8·097	8·095	8·094	8·092	8·091	8·089	8·087	8·085	8·082	1
2	8·620	8·618	8·617	8·615	8·613	8·611	8·609	8·606	8·604	8·601	2
3	9·113	9·112	9·110	9·108	9·106	9·103	9·100	9·098	9·095	9·091	3
4	9·580	9·578	9·576	9·574	9·571	9·568	9·565	9·562	9·558	9·555	4
15	10·022	10·019	10·017	10·014	10·011	10·008	10·004	10·001	9·997	9·992	15
6	10·439	10·437	10·434	10·431	10·428	10·424	10·420	10·415	10·411	10·406	6
7	10·834	10·831	10·828	10·825	10·821	10·817	10·812	10·807	10·802	10·796	7
8	11·208	11·204	11·201	11·197	11·193	11·188	11·182	11·177	11·171	11·164	8
9	11·561	11·557	11·553	11·548	11·544	11·538	11·532	11·526	11·519	11·512	9
20	11·894	11·890	11·886	11·881	11·876	11·870	11·863	11·856	11·848	11·840	20
1	12·210	12·205	12·200	12·195	12·189	12·182	12·175	12·167	12·159	12·150	1
2	12·508	12·503	12·497	12·491	12·485	12·478	12·470	12·461	12·452	12·442	2
3	12·789	12·784	12·778	12·771	12·764	12·756	12·747	12·738	12·728	12·717	3
4	13·055	13·049	13·043	13·036	13·028	13·019	13·010	12·999	12·988	12·976	4
25	13·307	13·300	13·293	13·285	13·277	13·267	13·257	13·246	13·234	13·220	25
6	13·544	13·537	13·529	13·521	13·511	13·501	13·490	13·478	13·465	13·450	6
7	13·768	13·760	13·752	13·743	13·733	13·722	13·709	13·696	13·682	13·667	7
8	13·980	13·971	13·962	13·952	13·941	13·930	13·916	13·902	13·887	13·870	8
9	14·179	14·170	14·160	14·150	14·138	14·125	14·111	14·096	14·079	14·061	9
30	14·367	14·358	14·347	14·336	14·323	14·309	14·294	14·278	14·260	14·241	30
1	14·545	14·535	14·523	14·511	14·498	14·483	14·466	14·449	14·430	14·409	1
2	14·712	14·701	14·689	14·676	14·662	14·646	14·628	14·610	14·589	14·567	2
3	14·870	14·858	14·845	14·831	14·816	14·799	14·780	14·760	14·739	14·715	3
4	15·018	15·006	14·992	14·977	14·961	14·943	14·923	14·902	14·879	14·854	4
35	15·158	15·145	15·130	15·114	15·097	15·078	15·057	15·034	15·010	14·983	35
6	15·290	15·275	15·260	15·243	15·225	15·204	15·182	15·158	15·132	15·104	6
7	15·413	15·398	15·382	15·364	15·344	15·323	15·299	15·274	15·247	15·217	7
8	15·530	15·514	15·496	15·477	15·457	15·434	15·409	15·382	15·353	15·322	8
9	15·639	15·622	15·603	15·583	15·562	15·538	15·511	15·483	15·453	15·420	9
40	15·741	15·723	15·704	15·683	15·660	15·635	15·607	15·577	15·545	15.510	40
1	15·837	15·818	15·798	15·775	15·751	15·725	15·696	15·665	15·631	15·594	1
2	15·927	15·907	15·886	15·862	15·837	15·809	15·778	15·746	15·710	15·672	2
3	16·011	15·990	15·968	15·943	15·917	15·887	15·855	15·821	15·784	15·744	3
4	16·090	16·068	16·044	16·018	15·991	15·960	15·926	15·890	15·851	15·809	4
45	16·164	16·141	16·116	16·088	16·059	16·027	15·992	15·955	15·914	15·870	45
6	16·232	16·208	16·182	16·154	16·123	16·089	16·053	16·014	15·971	15·925	6
7	16·296	16·271	16·244	16·214	16·182	16·147	16·109	16·068	16·024	15·976	7
8	16·355	16·329	16·301	16·269	16·236	16·200	16·160	16·117	16·071	16·022	8
9	16·411	16·383	16·353	16·321	16·286	16·248	16·207	16·163	16·115	16·064	9
	10	11	12	13	14	15	16	17	18	19	

O[M] $a_{[x]\overline{n}|}$ **5 per-cent.**

Duration	20	21	22	23	24	25	26	27	28	29	Duration
	16·262	16·179	16·092	16·002	15·907	15·810	15·707	15·601	15·490	15·375	
0	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000	0
1	·950	·950	·950	·950	·950	·950	·950	·950	·950	·949	1
2	1·851	1·850	1·850	1·850	1·850	1·850	1·850	1·850	1·849	1·849	2
3	2·704	2·704	2·704	2·703	2·703	2·703	2·703	2·702	2·702	2·701	3
4	3·513	3·512	3·512	3·511	3·511	3·510	3·510	3·509	3·508	3·508	4
5	4·278	4·278	4·277	4·276	4·276	4·275	4·274	4·273	4·272	4·271	5
6	5·003	5·002	5·001	5·000	4·999	4·998	4·997	4·996	4·994	4·992	6
7	5·689	5·688	5·686	5·685	5·684	5·682	5·680	5·679	5·676	5·674	7
8	6·338	6·336	6·335	6·333	6·331	6·329	6·327	6·324	6·321	6·319	8
9	6·951	6·949	6·947	6·945	6·943	6·940	6·937	6·934	6·930	6·927	9
10	7·531	7·529	7·527	7·524	7·521	7·518	7·514	7·510	7·505	7·501	10
1	8·080	8·077	8·074	8·071	8·067	8·063	8·058	8·054	8·048	8·042	1
2	8·598	8·594	8·591	8·587	8·582	8·578	8·572	8·567	8·560	8·553	2
3	9·088	9·083	9·079	9·074	9·069	9·063	9·057	9·050	9·042	9·035	3
4	9·550	9·545	9·540	9·535	9·528	9·522	9·514	9·507	9·497	9·488	4
15	9·987	9·981	9·976	9·969	9·962	9·954	9·946	9·937	9·926	9·915	15
6	10·400	10·393	10·387	10·379	10·371	10·362	10·352	10·342	10·330	10·318	6
7	10·789	10·782	10·774	10·766	10·756	10·747	10·735	10·724	10·710	10·696	7
8	11·157	11·149	11·140	11·131	11·120	11·109	11·096	11·083	11·068	11·052	8
9	11·504	11·495	11·485	11·475	11·462	11·450	11·436	11·421	11·404	11·386	9
20	11·831	11·821	11·810	11·798	11·785	11·771	11·756	11·739	11·720	11·700	20
1	12·139	12·128	12·116	12·104	12·089	12·073	12·056	12·038	12·017	11·995	1
2	12·430	12·418	12·405	12·391	12·374	12·358	12·338	12·318	12·295	12·271	2
3	12·704	12·691	12·677	12·661	12·643	12·625	12·604	12·582	12·557	12·530	3
4	12·963	12·948	12·932	12·915	12·896	12·875	12·853	12·828	12·801	12·772	4
25	13·206	13·190	13·172	13·154	13·133	13·111	13·086	13·060	13·030	12·999	25
6	13·434	13·417	13·398	13·378	13·355	13·331	13·305	13·276	13·244	13·210	6
7	13·649	13·631	13·610	13·589	13·564	13·538	13·509	13·478	13·444	13·408	7
8	13·851	13·831	13·809	13·786	13·759	13·732	13·701	13·667	13·631	13·591	8
9	14·041	14·020	13·996	13·971	13·942	13·913	13·879	13·844	13·804	13·762	9
30	14·219	14·196	14·171	14·144	14·113	14·081	14·046	14·008	13·966	13·921	30
1	14·386	14·362	14·335	14·306	14·273	14·239	14·201	14·160	14·115	14·068	1
2	14·543	14·516	14·488	14·457	14·422	14·386	14·345	14·302	14·254	14·203	2
3	14·689	14·661	14·630	14·598	14·561	14·522	14·479	14·433	14·383	14·329	3
4	14·826	14·796	14·764	14·729	14·690	14·649	14·603	14·555	14·501	14·444	4
35	14·954	14·922	14·888	14·851	14·810	14·766	14·718	14·667	14·610	14·550	35
6	15·073	15·039	15·003	14·964	14·920	14·874	14·824	14·770	14·710	14·647	6
7	15·184	15·149	15·110	15·069	15·023	14·975	14·921	14·864	14·802	14·735	7
8	15·287	15·250	15·209	15·166	15·118	15·067	15·011	14·951	14·886	14·816	8
9	15·383	15·344	15·301	15·255	15·205	15·151	15·093	15·030	14·962	14·889	9
40	15·472	15·431	15·386	15·338	15·285	15·229	15·167	15·102	15·030	14·955	40
1	15·554	15·511	15·464	15·413	15·358	15·299	15·235	15·167	15·093	15·014	1
2	15·630	15·584	15·535	15·483	15·425	15·364	15·297	15·226	15·148	15·066	2
3	15·699	15·652	15·601	15·546	15·486	15·422	15·352	15·278	15·198	15·113	3
4	15·763	15·714	15·660	15·603	15·541	15·474	15·402	15·326	15·243	15·155	4
45	15·822	15·770	15·715	15·655	15·590	15·521	15·447	15·367	15·282	15·191	45
6	15·875	15·822	15·764	15·702	15·635	15·564	15·486	15·404	15·316	15·223	6
7	15·924	15·868	15·808	15·744	15·675	15·601	15·521	15·437	15·346	15·250	7
8	15·968	15·910	15·848	15·782	15·710	15·634	15·552	15·465	15·372	15·274	8
9	16·008	15·948	15·884	15·815	15·741	15·663	15·579	15·490	15·394	15·294	9
	20	21	22	23	24	25	26	27	28	29	

0[M] $a_{[x]\overline{n}|}$ **5 per-cent.**

Duration	30	31	32	33	34	35	36	37	38	39	Duration
	15·255	15·130	15·001	14·866	14·728	14·583	14·434	14·279	14·120	13·955	
0	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000	0
1	·949	·949	·949	·949	·949	·949	·949	·949	·949	·948	1
2	1·849	1·849	1·849	1·848	1·848	1·848	1·847	1·847	1·846	1·846	2
3	2·701	2·700	2·700	2·699	2·699	2·698	2·697	2·696	2·695	2·694	3
4	3·507	3·506	3·505	3·504	3·503	3·502	3·500	3·499	3·497	3·495	4
5	4·269	4·268	4·267	4·265	4·263	4·261	4·259	4·257	4·254	4·252	5
6	4·990	4·989	4·987	4·984	4·982	4·979	4·976	4·972	4·969	4·965	6
7	5·672	5·669	5·666	5·663	5·660	5·656	5·651	5·647	5·642	5·636	7
8	6·315	6·312	6·308	6·303	6·299	6·294	6·288	6·282	6·276	6·269	8
9	6·923	6·918	6·913	6·908	6·902	6·896	6·888	6·881	6·872	6·863	9
10	7·496	7·490	7·484	7·477	7·470	7·462	7·453	7·443	7·433	7·421	10
1	8·036	8·029	8·022	8·013	8·005	7·995	7·984	7·972	7·959	7·945	1
2	8·546	8·537	8·528	8·518	8·508	8·496	8·483	8·469	8·454	8·437	2
3	9·026	9·016	9·005	8·993	8·981	8·967	8·951	8·935	8·917	8·897	3
4	9·478	9·466	9·454	9·440	9·425	9·409	9·391	9·372	9·351	9·328	4
15	9·903	9·890	9·876	9·860	9·843	9·824	9·803	9·781	9·757	9·731	15
6	10·304	10·288	10·272	10·254	10·235	10·213	10·190	10·164	10·137	10.107	6
7	10·680	10·663	10·645	10·624	10·602	10·578	10·551	10·523	10·491	10·458	7
8	11·034	11·015	10·994	10·971	10·946	10·919	10·889	10·857	10·822	10·784	8
9	11·366	11·345	11·322	11·296	11·268	11·238	11·205	11·169	11·130	11·088	9
20	11·678	11·654	11·629	11·600	11·569	11·536	11·499	11·459	11·416	11·370	20
1	11·971	11·944	11·916	11·884	11·851	11·814	11·773	11·730	11·682	11·631	1
2	12·245	12·216	12·184	12·150	12·113	12·072	12·028	11·980	11·929	11·873	2
3	12·501	12·469	12·435	12·397	12·357	12·313	12·264	12·213	12·156	12·095	3
4	12·741	12·706	12·669	12·628	12·584	12·536	12·484	12·427	12·366	12·301	4
25	12·964	12·927	12·887	12·842	12·795	12·743	12·686	12·626	12·560	12·489	25
6	13·173	13·133	13·089	13·041	12·990	12·934	12·873	12·808	12·737	12·661	6
7	13·368	13·324	13·277	13·226	13·171	13·111	13·045	12·976	12·900	12·819	7
8	13·548	13·502	13·452	13·396	13·337	13·273	13·203	13·129	13·048	12·962	8
9	13·716	13·666	13·613	13·553	13·491	13·422	13·348	13·269	13·183	13·091	9
30	13·872	13·818	13·761	13·698	13·632	13·559	13·480	13·396	13·305	13·208	30
1	14·015	13·959	13·898	13·831	13·761	13·683	13·600	13·511	13·416	13·313	1
2	14·148	14·088	14·024	13·953	13·878	13·797	13·709	13·615	13·514	13·407	2
3	14·270	14·207	14·138	14·064	13·985	13·899	13·807	13·708	13·603	13·490	3
4	14·382	14·315	14·243	14·165	14·082	13·992	13·895	13·792	13·682	13·564	4
35	14·485	14·414	14·339	14·256	14·169	14·075	13·974	13·866	13·751	13·628	35
6	14·578	14·504	14·425	14·339	14·248	14·149	14·044	13·932	13·812	13·685	6
7	14·663	14·586	14·503	14·413	14·318	14·215	14·106	13·989	13·865	13·734	7
8	14·741	14·659	14·573	14·479	14·380	14·274	14·160	14·039	13·911	13·775	8
9	14·810	14·726	14·636	14·538	14·435	14·325	14·207	14·083	13·950	13·811	9
40	14·873	14·785	14·691	14·590	14·484	14·370	14·248	14·120	13·984	13·841	40
1	14·928	14·837	14·740	14·636	14·526	14·408	14·283	14·151	14·012	13·866	1
2	14.978	14·884	14·783	14·676	14·562	14·441	14·313	14·178	14·036	13·886	2
3	15·022	14·925	14·821	14·710	14·594	14·469	14·338	14·200	14·055	13·903	3
4	15·060	14·960	14·853	14·739	14·620	14·493	14·359	14·218	14·070	13·916	4
45	15·094	14·990	14·881	14·765	14·642	14·513	14·376	14·233	14·083	13·926	45
6	15·123	15·017	14·905	14·786	14·661	14·529	14·390	14·245	14·093	13·934	6
7	15·148	15·039	14·925	14·803	14·676	14·542	14·401	14·254	14·100	13·940	7
8	15·169	15·058	14·942	14·818	14·689	14·552	14·410	14·261	14·106	13·945	8
9	15·187	15·074	14·955	14·829	14·699	14·561	14·417	14·266	14·110	13·948	9
	30	31	32	33	34	35	36	37	38	39	

O[M] $a_{[x]\overline{n}|}$ **5 per-cent.**

Duration	40	41	42	43	44	45	46	47	48	49	Duration
	13·784	13·610	13·429	13·242	13·051	12·854	12·652	12·445	12·233	12·017	
0	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000	0
1	·948	·948	·948	·948	·947	·947	·947	·946	·946	·946	1
2	1·845	1·845	1·844	1·843	1·843	1·842	1·841	1·840	1·839	1·837	2
3	2·693	2·692	2·691	2·689	2·687	2·686	2·684	2·682	2·679	2·677	3
4	3·493	3·491	3·489	3·486	3·484	3·481	3·477	3·474	3·470	3·465	4
5	4·248	4·245	4·242	4·238	4·233	4·229	4·224	4·218	4·212	4·205	5
6	4·960	4·956	4·950	4·945	4·938	4·932	4·924	4·916	4·907	4·898	6
7	5·630	5·624	5·617	5·609	5·600	5·591	5·581	5·570	5·558	5·545	7
8	6·261	6·252	6·243	6·233	6·221	6·209	6·196	6·182	6·166	6·149	8
9	6·853	6·842	6 830	6·817	6·803	6·788	6·771	6·752	6·732	6·711	9
10	7·409	7·396	7·381	7·365	7·347	7·328	7·307	7·284	7·259	7·232	10
1	7·930	7·914	7·896	7·876	7·855	7·831	7·806	7·778	7·748	7·716	1
2	8·418	8·399	8·378	8·354	8·328	8·300	8·270	8·237	8·202	8·163	2
3	8·875	8·853	8·827	8·799	8·769	8·737	8·701	8·662	8·621	8·575	3
4	9·303	9·276	9·247	9·214	9·179	9·142	9·100	9·056	9·007	8·955	4
15	9·702	9·671	9·637	9 600	9·560	9·517	9·470	9·418	9·363	9·303	15
6	10·074	10·039	10·001	9·958	9·913	9·864	9·810	9·752	9·690	9·622	6
7	10·420	10·381	10·338	10·290	10·239	10·184	10·124	10·059	9·989	9·914	7
8	10·743	10·699	10·650	10·597	10·540	10·478	10·412	10·339	10·262	10·178	8
9	11·042	10·993	10·939	10·880	10·817	10·749	10·675	10·596	10·510	10·418	9
20	11·319	11·265	11·205	11·141	11·071	10·996	10·915	10·828	10·735	10·634	20
1	11·575	11·516	11·451	11·380	11·304	11·222	11·134	11·039	10·938	10·829	1
2	11·812	11·747	11·676	11·599	11·516	11·428	11·332	11·229	11·120	11·002	2
3	12·029	11·959	11·882	11·799	11·709	11·614	11·511	11·400	11·282	11·156	3
4	12·229	12·153	12·070	11·980	11·884	11·781	11·671	11·553	11·427	11·293	4
25	12·412	12 330	12·241	12·145	12·042	11·932	11·814	11·689	11·555	11·412	25
6	12·579	12·491	12·396	12·293	12·184	12·067	11·942	11·809	11·667	11·517	6
7	12·730	12·637	12·536	12·427	12·311	12·187	12·055	11·914	11·765	11·607	7
8	12·868	12·769	12·661	12·546	12·423	12·292	12·153	12·006	11·849	11·684	8
9	12·992	12·887	12·773	12·652	12·523	12·385	12·239	12·085	11·922	11·750	9
30	13·103	12 992	12·873	12·745	12·610	12·466	12·314	12·153	11·984	11·805	30
1	13·203	13·086	12·961	12·828	12·686	12·536	12·378	12·211	12·035	11·851	1
2	13·291	13·169	13·039	12·899	12·752	12·596	12·432	12·259	12·079	11·889	2
3	13·369	13·242	13·106	12·961	12·808	12·647	12·478	12·300	12·114	11·920	3
4	13·438	13·305	13·164	13·014	12·856	12·690	12·516	12·333	12·143	11·944	4
35	13·498	13·360	13·214	13·059	12·897	12·726	12·547	12·360	12·166	11·963	35
6	13·549	13·407	13·257	13·097	12·930	12 755	12·572	12·382	12·184	11·978	6
7	13·594	13·447	13·292	13·129	12·958	12 779	12·593	12·399	12·198	11·990	7
8	13·631	13·481	13·322	13·155	12·980	12·798	12·608	12·412	12·208	11·998	8
9	13·663	13·509	13·347	13·176	12·998	12·813	12·621	12·422	12·216	12·004	9
40	13·690	13·532	13·366	13·193	13·012	12·824	12·630	12·429	12·222	12 008	40
1	13·711	13·551	13·382	13·206	13·023	12·833	12·637	12 434	12·226	12·011	1
2	13·729	13·566	13·395	13·216	13·031	12·840	12·642	12·438	12·229	12·013	2
3	13·743	13·577	13·404	13·224	13·037	12·844	12·646	12·441	12·230	12·015	3
4	13·754	13·586	13·412	13·230	13·042	12·848	12·648	12·442	12·232	12·015	4
45	13·762	13·593	13·417	13·234	13·045	12·850	12·650	12·444	12·232	12·016	45
6	13·769	13·598	13·421	13·237	13 047	12·852	12·651	12·444	12·233	12·016	6
7	13·774	13·602	13·424	13·239	13·048	12·853	12·651	12·445	12·233	12·016	7
8	13·777	13·605	13·426	13·240	13·049	12·853	12 652	12·445	12·233	12·016	8
9	13·780	13·606	13·427	13·241	13·050	12·854	12·652	12·445	12·233	12·016	9
	40	41	42	43	44	45	46	47	48	49	

O[M] $a_{[x]\,\overline{n}|}$ **5 per-cent.**

Duration	50	51	52	53	54	55	56	57	58	59	Duration
	11·795	11·569	11·338	11·104	10·865	10·623	10·377	10·128	9·876	9·622	
0	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000	0
1	·945	·945	·944	·944	·943	·912	·942	·941	·940	·939	1
2	1·836	1·835	1·833	1·831	1·829	1·827	1·825	1·823	1·820	1·817	2
3	2·674	2·671	2·668	2·664	2·660	2 656	2·651	2·646	2·640	2·634	3
4	3·461	3·455	3·450	3·444	3·437	3·429	3·421	3·413	3·403	3·393	4
5	4·198	4·190	4·181	4·171	4·161	4·150	4·138	4·124	4·109	4·094	5
6	4·887	4·876	4·863	4·850	4·835	4·819	4·801	4·782	4·761	4·739	6
7	5·531	5·515	5·498	5·480	5·460	5·438	5·414	5·388	5·360	5·330	7
8	6·130	6·110	6·088	6·063	6·037	6·009	5·978	5·944	5 908	5·869	8
9	6·687	6·661	6·633	6·603	6·570	6·534	6·495	6·453	6·407	6·358	9
10	7·203	7·171	7·137	7·099	7·059	7·014	6·967	6·915	6·860	6·800	10
1	7·680	7·642	7 600	7·555	7·506	7·453	7·396	7·334	7·268	7·196	1
2	8·121	8·075	8·026	7·972	7·915	7·852	7·785	7·712	7·635	7·551	2
3	8·526	8·473	8·415	8·353	8·286	8·213	8·136	8 052	7·962	7·866	3
4	8·898	8·837	8·771	8·699	8·622	8·539	8·451	8·355	8·253	8·144	4
15	9·239	9·169	9·094	9·013	8 926	8·832	8·732	8·625	8·510	8·388	15
6	9·550	9·471	9·387	9·296	9·198	9·094	8·982	8·863	8·736	8·601	6
7	9 832	9·745	9·651	9 550	9·442	9·326	9·203	9·072	8 933	8·786	7
8	10·088	9·992	9·888	9·777	9·658	9·532	9·398	9·254	9·103	8·944	8
9	10·319	10·213	10·100	9·979	9·850	9·712	9·567	9·412	9·249	9·078	9
20	10·527	10·411	10·288	10·157	10·018	9 869	9·713	9·547	9·373	9·191	20
1	10·712	10·587	10·455	10·314	10·164	10·005	9·838	9·662	9·478	9·285	1
2	10·877	10·743	10·601	10·450	10·290	10·122	9·945	9·759	9·564	9·362	2
3	11·022	10·880	10·728	10·568	10·399	10·221	10·035	9·839	9·636	9·424	3
4	11·150	10·999	10·838	10·669	10·491	10·304	10·109	9·905	9·694	9·474	4
25	11·261	11·101	10·933	10·755	10·569	10·374	10·170	9·959	9·740	9·514	25
6	11·358	11·190	11·013	10 827	10·633	10·431	10·220	10·002	9·776	9·544	6
7	11·440	11·265	11·081	10·888	10·686	10·477	10·260	10·035	9 804	9·567	7
8	11·511	11·328	11·137	10·937	10·729	10·514	10·291	10·061	9·826	9·584	8
9	11·570	11·380	11·183	10·977	10·764	10·543	10·315	10 081	9·841	9·597	9
30	11·619	11·423	11·220	11·009	10·791	10·565	10·333	10·096	9·853	9·605	30
1	11·659	11·458	11·250	11·034	10·812	10·582	10·347	10·106	9·861	9·611	1
2	11·692	11·486	11·274	11·054	10·827	10·595	10·356	10·114	9·867	9·616	2
3	11·718	11·508	11·292	11·069	10·839	10·604	10·363	10·119	9·870	9·618	3
4	11·738	11·525	11·306	11·080	10·848	10·610	10·368	10 122	9·873	9·620	4
35	11·754	11·538	11·316	11·087	10·854	10·615	10·371	10·125	9·874	9·621	35
6	11·766	11·548	11·323	11·093	10·858	10 618	10·374	10·126	9·875	9·621	6
7	11·775	11·555	11·329	11·097	10·861	10·620	10·375	10·127	9·876	9·622	7
8	11·782	11·560	11·332	11·100	10·862	10·621	10·376	10·127	9·876	9·622	8
9	11·786	11·563	11·335	11·101	10·864	10·622	10·376	10·128	9·876	9·622	9
40	11·790	11·565	11·336	11·102	10·864	10·622	10·376	10·128	9·876	9·622	40
1	11·792	11·567	11·337	11·103	10·865	10·622	10·376	10·128	9·876	9·622	1
2	11·793	11·568	11·338	11·103	10·865	10 623	10·377	10·128	9·876	9·622	2
3	11·794	11·568	11·338	11·104	10·865	10·623	10·377	10·128	9 876	9·622	3
4	11·794	11·569	11·338	11·104	10·865	10·623	10·377	10·128	9·876	9·622	4
45	11·795	11·569	11·338	11·104	10·865	10·623	10·377	10·128	9·876	9·622	45
6	11·795	11·569	11·338	11·104	10·865	10·623	10·377	10·128	9·876	9·622	6
7	11·795	11·569	11·338	11·104	10·865	10·623	10·377	10·128	9·876	...	7
8	11·795	11·569	11·338	11·104	10·865	10·623	10·377	10·128	...	...	8
9	11·795	11·569	11·338	11·104	10·865	10·623	10·377	...	...	...	9
	50	51	52	53	54	55	56	57	58	59	

O[M] $a_{[x]\overline{n}|}$ **5 per-cent.**

Duration	60	61	62	63	64	65	66	67	68	69	Duration
	9·366	9·107	8·847	8·587	8·325	8·063	7·802	7·541	7·281	7·022	
0	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000	0
1	·938	·937	·936	·934	·933	·931	·930	·928	·926	·923	1
2	1·814	1·810	1·806	1·802	1·798	1·793	1·787	1·781	1·775	1·768	2
3	2·628	2·621	2·613	2·604	2·595	2·585	2·573	2·561	2·548	2·534	3
4	3·381	3·369	3·356	3·341	3·325	3·308	3·289	3·269	3·247	3·223	4
5	4·076	4·058	4·037	4·015	3·991	3·965	3·936	3·906	3·873	3·837	5
6	4·714	4·688	4·658	4·627	4·593	4·557	4·517	4·474	4·428	4·378	6
7	5·297	5·261	5·222	5·180	5·135	5·086	5·034	4·977	4·917	4·851	7
8	5·826	5·780	5·730	5·677	5·619	5·557	5·490	5·418	5·342	5·259	8
9	6·305	6·248	6·186	6·119	6·048	5·971	5·889	5·801	5·708	5·608	9
10	6·735	6·666	6·590	6·510	6·424	6·332	6·234	6·130	6·019	5·901	10
1	7·119	7·037	6·948	6·853	6·752	6·645	6·530	6·409	6·281	6·145	1
2	7·461	7·365	7·262	7·153	7·036	6·913	6·782	6·644	6·498	6·345	2
3	7·763	7·653	7·536	7·411	7·280	7·140	6·993	6·839	6·677	6·508	3
4	8·028	7·904	7·773	7·633	7·487	7·332	7·169	7·000	6·822	6·638	4
15	8·259	8·121	7·976	7·822	7·661	7·491	7·314	7·130	6·938	6·740	15
6	8·459	8·308	8·148	7·981	7·805	7·622	7·431	7·234	7·030	6·819	6
7	8·630	8·466	8·293	8·113	7·924	7·728	7·525	7·315	7·100	6·879	7
8	8·776	8·599	8·414	8·221	8·021	7·813	7·599	7·379	7·154	6·923	8
9	8·898	8·710	8·513	8·309	8·098	7·880	7·656	7·427	7·193	6·956	9
20	9·000	8·801	8·593	8·379	8·158	7·931	7·699	7·462	7·222	6·979	20
1	9·083	8·874	8·658	8·434	8·205	7·970	7·731	7·488	7·243	6·994	1
2	9·151	8·933	8·708	8·477	8·241	7·999	7·754	7·506	7·257	7·005	2
3	9·205	8·980	8·747	8·509	8·267	8·020	7·771	7·519	7·266	7·012	3
4	9·248	9·016	8·777	8·534	8·286	8·035	7·782	7·527	7·272	7·016	4
25	9·281	9·042	8·799	8·551	8·300	8·046	7·790	7·533	7·276	7·018	25
6	9·306	9·063	8·815	8·563	8·309	8·053	7·795	7·536	7·278	7·020	6
7	9·325	9·077	8·826	8·572	8·315	8·057	7·798	7·538	7·279	7·021	7
8	9·338	9·088	8·834	8·578	8·319	8·060	7·800	7·540	7·280	7·021	8
9	9·348	9·095	8·839	8·581	8·322	8·061	7·801	7·540	7·281	7·021	9
30	9·354	9·099	8·843	8·584	8·324	8·062	7·801	7·541	7·281	7·022	30
1	9·359	9·103	8·845	8·585	8·324	8·063	7·802	7·541	7·281	7·022	1
2	9·362	9·105	8·846	8·586	8·325	8·063	7·802	7·541	7·281	7·022	2
3	9·363	9·106	8·847	8·586	8·325	8·063	7·802	7·541	7·281	7·022	3
4	9·364	9·106	8·847	8·586	8·325	8·063	7·802	7·541	7·281	7·022	4
35	9·365	9·107	8·847	8·587	8·325	8·063	7·802	7·541	7·281	7·022	35
6	9·365	9·107	8·847	8·587	8·325	8·063	7·802	7·541	7·281	7·022	6
7	9·365	9·107	8·847	8·587	8·325	8·063	7·802	7·541	7·281		
8	9·366	9·107	8·847	8·587	8·325	8·063	7·802	7·541		47	
9	9·366	9·107	8·847	8·587	8·325	8·063	7·802		48	12·445	
40	9·366	9·107	8·847	8·587	8·325	8·063		49	12·233		
1	9·366	9·107	8·847	8·587	8·325		50	12·017		12·445	58
2	9·366	9·107	8·847	8·587		51	11·795		12·233	12·445	7
3	9·366	9·107	8·847		52	11·569		12·017	12·233	12·445	6
4	9·366	9·107		53	11·338		11·795	12·017	12·233	12·445	5
45	9·366		54	11·104		11·569	11·795	12·017	12·233	12·445	54
	...	55	10·865		11·338	11·569	11·795	12·017	12·233	12·445	3
	...	10·623		11·104	11·338	11·569	11·795	12·017	12·233	12·445	2
	...		10·865	11·104	11·338	11·569	11·795	12·017	12·233	12·445	1
	...	10·623	10·865	11·104	11·338	11·569	11·795	12·016	12·233	12·445	0
	60	55	54	53	52	51	50	49	48	47	

O[M] $a_{[x]\overline{n}|}$ **5 per-cent.**

Duration	70	71	72	73	74	75	Duration
	6·765	6·510	6·257	6·008	5·762	5·519	
0	·000	·000	·000	·000	·000	·000	0
1	·921	·918	·915	·912	·908	·904	1
2	1·760	1·752	1·743	1·733	1·722	1·711	2
3	2·519	2·502	2·483	2·464	2·442	2·419	3
4	3·197	3·169	3·139	3·106	3·071	3·033	4
5	3·798	3·756	3·712	3·663	3·612	3·556	5
6	4·325	4·267	4·206	4·140	4·070	3·995	6
7	4·781	4·707	4·627	4·542	4·452	4·356	7
8	5·172	5·079	4·980	4·875	4·765	4·649	8
9	5·502	5·390	5·272	5·147	5·017	4·880	9
10	5·777	5·646	5·509	5·365	5·215	5·059	10
1	6·003	5·854	5·698	5·536	5·368	5·195	1
2	6·186	6·020	5·847	5·669	5·485	5·296	2
3	6·333	6·150	5·962	5·769	5·571	5·369	3
4	6·448	6·251	6·049	5·843	5·634	5·421	4
15	6·537	6·328	6·114	5·898	5·678	5·457	15
6	6·604	6·384	6·161	5·936	5·709	5·481	6
7	6·654	6·426	6·195	5·962	5·729	5·496	7
8	6·691	6·455	6·218	5·980	5·742	5·506	8
9	6·716	6·475	6·233	5·991	5·751	5·512	9
20	6·734	6·489	6·243	5·998	5·756	5·515	20
1	6·746	6·497	6·249	6·003	5·758	5·517	1
2	6·754	6·503	6·253	6·005	5·760	5·518	2
3	6·759	6·506	6·255	6·007	5·761	5·518	3
4	6·761	6·508	6·256	6·007	5·761	5·519	4
25	6·763	6·509	6·257	6·008	5·761	5·519	25
6	6·764	6·509	6·257	6·008	5·762	5·519	6
7	6·764	6·510	6·257	6·008	5·762	5·519	7
8	6·765	6·510	6·257	6·008	5·762	5·519	
9	6·765	6·510	6·257	6·008	5·762	5·519	
30	6·765	6·510	6·257	6·008	5·762	5·519	
1	6·765	6·510	6·257	6·008	5·762		
2	6·765	6·510	6·257	6·008			
3	6·765	6·510	6·257				
4	6·765	6·510					
35	6·765						

Duration	46	45	44	43	42	41	40	39	38	37	Duration
	12·652	12·854	13·051	13·242	13·429	13·610	13·784	13·955	14·120	14·279	
										14·279	68
									14·120	14·279	7
								13·955	14·120	14·279	6
							13·784	13·955	14·120	14·279	5
						13·610	13·784	13·955	14·120	14·279	64
					13·429	13·610	13·784	13·955	14·120	14·279	3
				13·242	13·429	13·610	13·784	13·955	14·120	14·279	2
			13·051	13·242	13·429	13·610	13·784	13·955	14·120	14·279	1
		12·854	13·051	13·242	13·429	13·610	13·784	13·955	14·120	14·279	0
59	12·652	12·854	13·051	13·242	13·429	13·610	13·784	13·955	14·120	14·279	59
8	12·652	12·854	13·051	13·242	13·429	13·610	13·784	13·955	14·120	14·279	8
7	12·652	12·854	13·051	13·242	13·429	13·610	13·784	13·955	14·120	14·279	7
6	12·652	12·854	13·051	13·242	13·429	13·610	13·784	13·955	14·119	14·279	6
5	12·652	12·854	13·051	13·242	13·429	13·610	13·784	13·955	14·119	14·278	5
54	12·652	12·854	13·051	13·242	13·429	13·609	13·784	13·954	14·119	14·278	54
3	12·652	12·854	13·051	13·242	13·429	13·609	13·784	13·954	14·118	14·277	3
2	12·652	12·854	13·051	13·242	13·429	13·609	13·783	13·953	14·117	14·275	2
1	12·652	12·854	13·051	13·242	13·428	13·608	13·782	13·952	14·116	14·273	1
0	12·652	12·854	13·050	13·242	13·428	13·608	13·781	13·951	14·113	14·270	0
	46	45	44	43	42	41	40	39	38	37	

O[M] $a_{[x]\overline{n}|}$ **5 per-cent.**

Duration	36	35	34	33	32	31	30	29	28	27	Duration
	14·434	14·583	14·728	14·866	15·001	15·130	15·255	15·375	15·490	15·601	
										15·601	78
									15·490	15·601	7
								15·375	15·490	15·601	6
							15·255	15·375	15·490	15·601	5
						15·130	15·255	15·375	15·490	15·601	74
					15·001	15·130	15·255	15·375	15·490	15·601	3
				14·866	15·001	15·130	15·255	15·375	15·490	15·601	2
			14·728	14·866	15·001	15·130	15·255	15·375	15·490	15·601	1
		14·583	14·728	14·866	15·001	15·130	15·255	15·375	15·490	15·601	0
69	14·434	14·583	14·728	14·866	15·001	15·130	15·255	15·375	15·490	15·601	69
8	14·434	14·583	14·728	14·866	15·001	15·130	15·255	15·375	15·490	15·601	8
7	14·434	14·583	14·728	14·866	15·001	15·130	15·255	15·375	15·489	15·601	7
6	14·434	14·583	14·728	14·866	15·001	15·130	15·255	15·374	15·489	15·601	6
5	14·434	14·583	14·728	14·866	15·001	15·130	15·255	15·374	15·489	15·600	5
64	14·434	14·583	14·728	14·866	15·001	15·130	15·254	15·374	15·489	15·600	64
3	14·434	14·583	14·728	14·866	15·001	15·130	15·254	15·374	15·489	15·600	3
2	14·434	14·583	14·728	14·866	15·001	15·130	15·254	15·374	15·488	15·599	2
1	14·434	14·583	14·728	14·866	15·001	15·129	15·254	15·373	15·487	15·598	1
0	14·434	14·583	14·728	14·866	15·000	15·129	15·253	15·372	15·486	15·596	0
59	14·434	14·583	14·727	14·866	15·000	15·128	15·252	15·371	15·484	15·594	59
8	14·434	14·583	14·727	14·865	14·999	15·127	15·251	15·369	15·482	15·591	8
7	14·433	14·583	14·726	14·864	14·998	15·126	15·249	15·366	15·479	15·587	7
6	14·433	14·582	14·726	14·863	14·997	15·124	15·246	15·363	15·474	15·581	6
5	14·432	14·581	14·724	14·861	14·994	15·121	15·242	15·358	15·469	15·575	5
54	14·431	14·580	14·723	14·859	14·991	15·117	15·238	15·353	15·462	15·566	54
3	14·430	14·578	14·720	14·856	14·987	15·112	15·231	15·345	15·453	15·556	3
2	14·428	14·575	14·717	14·852	14·982	15·105	15·224	15·336	15·442	15·544	2
1	14·425	14·572	14·712	14·846	14·975	15·097	15·214	15·324	15·429	15·529	1
0	14·422	14·567	14·706	14·839	14·966	15·087	15·202	15·310	15·413	15·511	0
	36	35	34	33	32	31	30	29	28	27	

O[M] $a_{[x]\overline{n}|}$ **5 per-cent.**

Duration	26	25	24	23	22	21	20	19	18	17	Duration
	15·707	15·810	15·907	16·002	16·092	16·179	16·262	16·341	16·417	16·490	
										16·490	88
									16·417	16·490	7
								16·341	16·417	16·490	6
							16·262	16·341	16·417	16·490	5
						16·179	16·262	16·341	16·417	16·490	84
					16·092	16·179	16·262	16·341	16·417	16·490	3
				16·002	16·092	16·179	16·262	16·341	16·417	16·490	2
			15·907	16·002	16·092	16·179	16·262	16·341	16·417	16·490	1
		15·810	15·907	16·002	16·092	16·179	16·262	16·341	16·417	16·490	0
	15·707	15·810	15·907	16·002	16·092	16·179	16·262	16·341	16·417	16·490	79
	15·707	15·810	15·907	16·002	16·092	16·179	16·262	16·341	16·417	16·490	8
	15·707	15·810	15·907	16·002	16·092	16·179	16·262	16·341	16·417	16·490	7
	15·707	15·810	15·907	16·002	16·092	16·179	16·262	16·341	16·417	16·489	6
	15·707	15·810	15·907	16·002	16·092	16·179	16·261	16·341	16·417	16·489	5
	15·707	15·810	15·907	16·002	16·092	16·179	16·261	16·341	16·417	16·489	74
	15·707	15·810	15·907	16·002	16·092	16·179	16·261	16·341	16·416	16·489	3
	15·707	15·810	15·907	16·002	16·092	16·178	16·261	16·341	16·416	16·488	2
	15·707	15·810	15·907	16·002	16·092	16·178	16·261	16·340	16·416	16·488	1
	15·707	15·810	15·907	16·002	16·092	16·178	16·261	16·340	16·415	16·487	0
	15·707	15·810	15·907	16·002	16·091	16·178	16·260	16·339	16·414	16·485	69
	15·707	15·809	15·907	16·001	16·091	16·177	16·259	16·338	16·413	16·484	8
	15·707	15·809	15·907	16·001	16·090	16·176	16·258	16·337	16·411	16·481	7
	15·707	15·809	15·906	16·000	16·090	16·175	16·257	16·335	16·408	16·478	6
	15·706	15·808	15·905	15·999	16·088	16·173	16·255	16·332	16·405	16·475	5
	15·706	15·808	15·905	15·998	16·087	16·171	16·252	16·329	16·401	16·470	64
	15·705	15·807	15·903	15·996	16·084	16·168	16·248	16·325	16·396	16·464	3
	15·704	15·805	15·901	15·994	16·081	16·165	16·244	16·319	16·390	16·457	2
	15·702	15·803	15·899	15·991	16·077	16·160	16·238	16·313	16·382	16·448	1
	15·700	15·800	15·895	15·986	16·072	16·154	16·231	16·305	16·373	16·438	0
	15·697	15·797	15·891	15·981	16·066	16·147	16·223	16·295	16·362	16·426	59
	15·694	15·792	15·885	15·974	16·058	16·138	16·213	16·284	16·350	16·412	8
	15·689	15·786	15·878	15·966	16·049	16·127	16·201	16·270	16·335	16·396	7
	15·682	15·779	15·869	15·956	16·037	16·114	16·186	16·255	16·318	16·377	6
	15·674	15·770	15·859	15·944	16·024	16·099	16·170	16·236	16·298	16·356	5
	15·665	15·758	15·846	15·930	16·008	16·082	16·151	16·216	16·276	16·332	54
	15·653	15·745	15·831	15·913	15·989	16·061	16·129	16·192	16·250	16·305	3
	15·639	15·729	15·813	15·894	15·968	16·038	16·104	16·165	16·222	16·275	2
	15·622	15·710	15·793	15·871	15·944	16·012	16·075	16·135	16·190	16·241	1
	15·602	15·688	15·769	15·845	15·916	15·982	16·043	16·101	16·154	16·204	0
	26	25	24	23	22	21	20	19	18	17	

O[M] $a_{[x]\overline{n}|}$ **5 per-cent.**

Duration	16	15	14	13	12	11	10	Duration
							16·916	
						16·863	16·916	95
					16·808	16·863	16·916	94
				16·750	16·808	16·863	16·916	3
			16·689	16·750	16·808	16·863	16·916	2
		16·626	16·689	16·750	16·808	16·863	16·916	1
	16·559	16·626	16·689	16·750	16·808	16·863	16·916	0
89	16·559	16·626	16·689	16·750	16·808	16·863	16·916	89
8	16·559	16·626	16·689	16·750	16·808	16·863	16·916	8
7	16·559	16·626	16·689	16·750	16·808	16·863	16·916	7
6	16·559	16·626	16·689	16·750	16·808	16·863	16·916	6
5	16·559	16·626	16·689	16·750	16·808	16·863	16·916	5
84	16·559	16·626	16·689	16·750	16·808	16·863	16·916	84
3	16·559	16·626	16·689	16·750	16·808	16·863	16·916	3
2	16·559	16·626	16·689	16·750	16·808	16·863	16·916	2
1	16·559	16·626	16·689	16·750	16·808	16·863	16·916	1
0	16·559	16·626	16·689	16·750	16·808	16·863	16·915	0
79	16·559	16·626	16·689	16·750	16·807	16·863	16·915	79
8	16·559	16·626	16·689	16·749	16·807	16·862	16·915	8
7	16·559	16·626	16·689	16·749	16·807	16·862	16·914	7
6	16·559	16·625	16·689	16·749	16·806	16·861	16·913	6
5	16·559	16·625	16·688	16·748	16·806	16·860	16·912	5
74	16·558	16·625	16·688	16·747	16·805	16·859	16·910	74
3	16·558	16·624	16·687	16·746	16·803	16·857	16·908	3
2	16·557	16·623	16·686	16·745	16·802	16·855	16·906	2
1	16·556	16·622	16·684	16·743	16·799	16·852	16·902	1
0	16·555	16·620	16·682	16·741	16·796	16·849	16·898	0
69	16·553	16·618	16·680	16.738	16·793	16·845	16·894	69
8	16·551	16·616	16·677	16·734	16·788	16·840	16·888	8
7	16·548	16·612	16·673	16·729	16·783	16·833	16·881	7
6	16·545	16·608	16·668	16·723	16·776	16·826	16·873	6
5	16·540	16·603	16·662	16·717	16·769	16·817	16·863	5
64	16·535	16·596	16·654	16·708	16·760	16·807	16·852	64
3	16·528	16·589	16·646	16·699	16·749	16·796	16·839	3
2	16·520	16·580	16·635	16·687	16·737	16·782	16·825	2
1	16·510	16·569	16·624	16·674	16·722	16·767	16·809	1
0	16·499	16·556	16·610	16·660	16·706	16·750	16·790	0
59	16·486	16·542	16·594	16·643	16·688	16·731	16·770	59
8	16·470	16·525	16·576	16·623	16·668	16·709	16·747	8
7	16·453	16·506	16·556	16·602	16·645	16·685	16·722	7
6	16·433	16·485	16·533	16·578	16·619	16·658	16·694	6
5	16·410	16·460	16·507	16·551	16·591	16·628	16·663	5
54	16·384	16·433	16·479	16·521	16·560	16·596	16·629	54
3	16·356	16·403	16·447	16·488	16·525	16·560	16·593	3
2	16·324	16·370	16·412	16·451	16·488	16·521	16·552	2
1	16·289	16·333	16·374	16·412	16·447	16·479	16·509	1
0	16·250	16·293	16·332	16·368	16·402	16·433	16·462	0
	16	15	14	13	12	11	10	

THE INSTITUTE OF ACTUARIES.

PROCEEDINGS OF THE INSTITUTE.—Session 1917-1918.

First Ordinary Meeting, 4 *December* 1917.

The President (Mr. S. G. Warner) in the Chair.

The President delivered an Inaugural Address.

Second Ordinary Meeting, 14 *January* 1918.

The President (Mr. S. G. Warner) in the Chair.

Mr. William Maxwell Gunn Wilson, F.F.A., was duly elected an Associate of the Institute.

A discussion on "The Financial Management of Life Assurance Companies" was opened by Mr. O. T. Falk, in which the following gentlemen also took part, namely, Sir Gerald Ryan, Sir Alfred Watson, Messrs. W. P. Phelps, G. Marks, A. G. Mackenzie, C. R. V. Coutts, J. Burn, and the President.

The Seventy-first Annual General Meeting, 3 *June* 1918.

The President (Mr. S. G. Warner) in the Chair.

The proceedings at the Annual General Meeting will be found on page 166.

REPORT, 1917–1918.

The Council have the pleasure to report to the Members upon the work of the Institute during the Session of 1917–1918, the seventieth year of its existence.

There has been a *decrease* of 36 in the total number of members, as compared with the previous year. At the end of the official year in which the Institute was incorporated by Royal Charter the number of Members was 434; twenty-two years later, at 31 March 1907, it was 956. Since that time the numbers have been as follows:

On 31 March	Fellows	Associates	Students	Corresponding Members	Total
1908	253	313	421	22	1,009
1909	254	325	400	19	998
1910	259	335	348	21	963
1911	267	339	308	20	934
1912	278	354	268	20	920
1913	282	355	252	19	908
1914	295	358	238	19	910
1915	304	361	263	17	945
1916	308	345	247	17	917
1917	303	344	231	18	896
1918	295	332	215	18	860

The following schedule shows the additions to, and the changes and losses in the membership which have occurred during the year ending 31 March last:

Schedule of Membership, 31 *March* 1918.

	Fellows	Associates	Students	Corresponding Members	Total
i. Number of Members in each class on 31 March 1917	303	344	231	18	896
ii. Withdrawals by					
(1) Death	8	11	6	...	41
(2) Resignation or otherwise	...	6	10	...	
	295	327	215	18	855
iii. Additions to Membership					
(1) By Election	...	2	...	...	5
(2) By Examination	...	...	...	...	
(3) By Re-instatement	...	2	1	...	
	295	331	216	18	860
iv. Transfers					
(1) By Examination: *from* Associates	...	...	...	...	...
to Fellows	...	...	...	...	...
	295	331	216	18	860
(2) By Examination: *from* Students	...	...	1	...	...
to Associates	...	1	...	...	...
v. Number of Members in each class on 31 March 1918	295	332	215	18	860

There are also 156 candidates admitted as Probationers, and 70 as Students conditionally on their passing Part I of the Examination. These are not included in the above Schedule of Membership. The numbers in these two classes since 31 March 1912 have been as follows:

On 31 March	Probationers	Conditional Students	On 31 March	Probationers	Conditional Students
1913	197	55	1916	172	73
1914	200	67	1917	173	67
1915	188	72	1918	156	70

The Council have, with great regret, to report the loss by death, since the last Annual Meeting, of eight Fellows, Messrs. H. W. Andras, D. A. Bumsted, G. D. Doucet, J. Graham, E. A. Newton, W. F. Somerville, G. Todd, and A. H. Turnbull; seven Associates, Messrs. F. S. Blake, G. A. Brown, R. A. Craig, G. S. Fielden, J. Hogg, E. Litchfield, and J. J. Stuckey; and five Students, Messrs. T. N. Askwith, H. J. Hammond, T. Holgate, B. Needell, and H. Orrell. Nine of these Members, namely, Captains F. S. Blake, H. J. Hammond, and J. Hogg, Lieutenants T. N. Askwith, G. D. Doucet, and G. S. Fielden, Sergeant B. Needell, and Privates

T. Holgate and H. Orrell have fallen in the service of their King and Country; as also have three Probationers of the Institute, Captain W. E. Smith, and Lieutenants W. Askham and S. G. Weatherdon. The Council have sent letters of sympathy to the relatives of all who have thus sacrificed their lives in the nation's cause.

Messrs. H. W. Andras and G. Todd had both filled the office of Vice-President, and had for many years been Members of the Council. Mr. Todd had also served the Institute in the capacity of Joint Honorary Secretary, and Mr. Andras rendered valuable assistance for many years as Joint Honorary Librarian.

The Annual Subscriptions and the Entrance Fees appearing in the Revenue Account amounted to £1,376. 0*s*. 6*d*., as compared with £1,476. 6*s*. 0*d*. in the previous year. The Income and Expenditure for the year were £1,812. 2*s*. 11*d*. and £1,609. 5*s*. 2*d*. respectively.

The number of Members and Probationers on the roll of service with the Army and Navy has, since last year, increased to 403. The Council have to deplore the loss of 47 who have been killed in action or died of wounds.

The position of candidates whose preparation for the Examinations has been affected by the War has been receiving the sympathetic consideration of the Council, and with the view of giving candidates every possible assistance, having regard to the special circumstances, the Council have decided for the present to revise the Syllabus so as to limit the amount of the necessary reading; to request the Board of Examiners to afford some guidance as to such reading; and to hold Examinations more frequently than has hitherto been the practice. The first Examinations under the new Syllabus will be held as soon as possible after the termination of the War.

The revised Regulations and Syllabus of Examinations have already been published in the April Number of the *Journal*.

Mr. Arthur James Cook, who died on the 25 February 1917, bequeathed to the Institute a share in the ultimate residue of his estate for the furtherance of the objects of the G. F. Hardy Memorial Fund, and in remembrance of his long connection with the Institute as an Associate. An intimation has been received to the effect that the Memorial Fund thus benefits to the extent of £135. 11*s*. 3*d*., and the Council desire to take this opportunity of placing on record their appreciation of the bequest.

The stock in hand of the Institute publications on 31 March was as follows:

No. of Copies	Description of Work
28,825	Parts of *Journal*.
730	Index to Vols. 1 to 40.
1,708	*Text-Book*, Part I (Revised Edition).
491	*Text-Book*, Part II (Second Edition).
633	Government Joint-Life Annuity Tables.
733	Select Life Tables.
33	A Short Collection of Actuarial Tables (New Edition).
922	Frequency-Curves and Correlation (W. P. Elderton).
172	Ditto ditto ditto (Addendum and Errata).
38 *in cloth* 2,314 *in paper*	Lectures on Finance and Law (Clare and Wood Hill).

[*Continued on page* 166.]

Dr. *Revenue Account for the*

1917. £ s. d.		1918. £ s. d.	£ s. d.
	Amount of Funds at the beginning of the year—		
8,190 4 8	General Fund (including Stock of Publications, other than *Journal*)	8,045 3 7	
421 5 5	Messenger Legacy Fund	433 18 2	
361 18 11	Brown Prize Fund	372 16 1	
767 15 9	G. F. Hardy Memorial Fund	793 15 3	
9,741 4 9			9,645 13 1
	Subscriptions—		
792 15 0	Fellows	751 16 0	
590 2 0	Associates	532 7 0	
157 10 0	Students	126 0 0	
36 4 6	Probationers	13 2 6	
1,576 11 6		1,423 5 6	
1 1 0	Fines on Reinstatement	3 3 0	
1,577 12 6		1,426 8 6	
	Less Waived and returned to Members and Probationers		
105 10 6	on Naval and Military Service	57 15 0	
1,472 2 0			1,368 13 6
	Entrance Fees—		
...	Associates	4 4 0	
2 2 0	Students	3 3 0	
2 2 0	*Probationers*	...	
4 4 0			7 7 0
125 16 3	Balance of Publications Account		76 19 11
	Dividends and Interest—		
245 1 1	General Fund	295 14 4	
12 12 9	Messenger Legacy Fund	13 0 4	
10 17 2	Brown Prize Fund	11 3 8	
25 19 6	G. F. Hardy Memorial Fund	39 4 2	
294 10 6			359 2 6
£11,637 17 6			£11,457 16 0

Publications Account for the

£ s. d.		£ s. d.
297 15 7	Stock (excluding *Journal*) at the beginning of the year	249 0 2
...	Cost of Reprint of Text-Book, Part II	105 12 8
12 2 4	Binding and Advertising	33 3 4
125 16 3	Balance	76 19 11
£435 14 2		£464 16 1

Balance Sheet,

£ s. d.	LIABILITIES.	£ s. d.	£ s. d.	£ s. d.
8,045 3 7	General Fund		8,184 13 2	
233 9 2	Messenger Legacy Fund	233 9 2		
200 9 0	Accumulated Dividends	213 9 4		
433 18 2			446 18 6	
200 0 0	Brown Prize Fund	200 0 0		
172 16 1	Accumulated Dividends	183 19 9		
372 16 1			383 19 9	
767 15 9	G. F. Hardy Memorial Fund	767 15 9		
25 19 6	Accumulated Dividends	65 3 8		
793 15 3			832 19 5	
				9,848 10 10
17 6 2	Sundry unpaid Accounts			20 5 11
£9,662 19 3				£9,868 16 9

year ending 31 *March* 1918. **Cr.**

1917. £ s. d.		£ s. d.	1918. £ s. d.
	Journal—		
220 9 5	Printing of Nos. 267, 268	260 3 4	
48 8 0	Editorial Expenses	43 2 6	
268 17 5		303 5 10	
112 16 9	*Less* Sales during the year	102 5 2	
156 0 8			201 0 8
19 8 6	Library—Binding, Purchases, &c.		25 8 3
41 14 9	Meetings		25 6 4
18 11 3	*Legal Charges*		...
600 0 0	Office Expenditure—Rent	600 0 0	
488 17 2	Salaries	520 9 0	
66 19 3	House expenses	49 17 2	
39 14 5	Fire and other Insurance	30 7 2	
69 4 9	Stationery and Printing	78 17 11	
23 17 9	Postage and Telegrams	20 5 0	
11 7 10	Sundries	57 13 8	
1,300 1 2			1,357 9 11
456 8 1	*Loss on Sale of Stock Exchange Securities*		...
9,645 13 1	Amount of Funds at the end of the year as per Balance Sheet		9,848 10 10
£11,637 17 6			£11,457 16 0

Examined and found correct, 25 *April* 1918.

GEORGE H. LAWTON,
E. W. HUMPHRY,
STANLEY HAZELL, } *Auditors.*

year ending 31 *March* 1918.

£ s. d.		£ s. d.
186 14 0	Sales (excluding *Journal*)	140 16 11
249 0 2	Stock (excluding *Journal*) at the end of the year	323 19 2
£435 14 2		£464 16 1

Examined and found correct, 25 *April* 1918.

GEORGE H. LAWTON,
E. W. HUMPHRY,
STANLEY HAZELL, } *Auditors.*

31 *March* 1918.

£ s. d.	ASSETS.	£ s. d.
1,875 0 0	£3,000 Natal 3 per-cent Inscribed Stock	1,875 0 0
726 5 0	£1,000 Dominion of Canada $3\frac{1}{2}$ per-cent Registered 1930–50 Stock	726 5 0
700 0 0	£1,000 New South Wales $3\frac{1}{2}$ per-cent Inscribed 1930–50 Stock	700 0 0
357 15 0	£600 Belgian Government 3 per-cent Sterling Loan of 1914	357 15 0
4,275 0 0	£4,500 5 per-cent War Stock, 1929–47	4,275 0 0
...	£400 National War Bonds, 4 per-cent, 1927	400 0 0
249 0 2	Stock of Publications (excluding *Journal*) in hand	323 19 2
200 0 0	*Cash on Deposit Account*	...
441 16 4	Cash on Current Account and in hand	355 18 10
70 7 0	Subscriptions in Arrear	87 3 0
767 15 9	£810. 5*s.* 8*d.* 5 per-cent War Stock, 1929–47—$4\frac{1}{2}$ *per-cent Converted*—(G. F. Hardy Fund)	767 15 9
£9,662 19 3		£9,868 16 9

The Stock Exchange Securities are taken at the values at which they stood on 31 March 1916, and at cost price in the case of any acquired since that date.

Examined and found correct, 25 *April* 1918.

GEORGE H. LAWTON,
E. W. HUMPHRY,
STANLEY HAZELL, } *Auditors.*

No. of Copies	Description of Work
1,526	Lectures on the Companies Acts (A. C. Clauson).
1,189	Lectures on the Law of Mortgage (W. G. Hayter).
703	Lectures on the Measurement of Groups and Series (A. L. Bowley).
1,400	Lectures on the Construction of Tables of Mortality, &c. (Sir G. F. Hardy, K.C.B.).
884	Lectures on Stock Exchange Investments (J. Burn).
1,500	Lectures on Friendly Society Finance (Sir A. W. Watson).
315	South African War Mortality (F. Schooling and E. A. Rusher).
248	Life Assurance Law (A. R. Barrand).
654	British Offices' Valuation Tables.
652	British Offices' $2\frac{3}{4}$ per-cent Temporary Annuity Values.
138	Transactions of the Second International Congress of Actuaries.
785	Index to Transactions of Seven International Actuarial Congresses.
1,500	Examination Questions, 1912–15.

7 *May* 1918.

PROCEEDINGS AT THE ANNUAL GENERAL MEETING.

The Seventy-First Annual General Meeting of the Institute of Actuaries was held in Staple Inn Hall, Holborn, on Monday, 4 June 1918, Mr. Samuel G. Warner (the President) in the Chair.

Mr. J. Douglas Watson (Honorary Secretary), read the notice convening the Meeting. The Minutes of the preceding General Meeting were read and confirmed and the Report and Accounts were taken as read.

The President, in moving the adoption of the Report and Accounts, said that there was not much in the Report to which special attention need be drawn. There was a slight decrease in the number of members, due to the present conditions, and the examinations were at a standstill. There was, he regretted to say, a long obituary list. The place of honour on that list, should, he thought, be given to those who had fallen in the service of their country. As they would have seen from the report the total number connected with the Institute who had entered military service was 403, and of those 47 had fallen on the field. Five had suffered that fate since he last addressed the Institute, and he would like to say something about one of them, Mr. H. T. Kay Robinson. Mr. Robinson became a Fellow of the Institute by examination in 1905, joined the army at the outbreak of war and rose to the rank of Lieut.-Colonel and was awarded the Distinguished Service Order and bar for conspicuous gallantry in the field. This, he thought, was a record of which the Institute should be proud. And the Institute was proud of all who had made this supreme sacrifice. They were too young to have gained that professional distinction of which many of them gave promise, but they had attained a distinction which nothing could take from them, a distinction which would live imperishably in the memories of their kindred and their friends.

There would be noticed in the report an expression of regret at the loss of a number of valued members of the profession. Mr. Henry Walsingham Andras was well known to all. He took a keen interest in the Institute, and though his failing health prevented him in recent years from taking an active part in its affairs he had served it as vice-president and librarian, and

was for 14 years a member of the Council. Everybody who knew him appreciated the geniality of his character, the value of his friendship and his service to the Institute. Following in a brief space of time came the death of Mr. George Todd. His actuarial career had much in common with that of Mr. Andras. They were contemporaries, both taking a great interest in the Institute's affairs, and held in high esteem by its members. Mr. Todd was for 23 years a member of the Council and for four years honorary secretary. The remaining two losses by death had a special, and what might be called a historic, interest. In the preamble to their Charter there was a reference to 15 gentlemen there named, of whom it was stated that "being members of a collective body established in the year 1848 under the name of the Actuaries' Club and exercising the calling or profession of actuary," they "are desirous of becoming entitled to such privileges as may arise from the incorporation of the members of the Institute of Actuaries under our Royal Charter." Those fifteen names represent what might be called the actuarial profession of the past; and by the deaths, within the last month or two, of Mr. Priestley and Mr. Stevens the last of them disappeared. A link with the past was thus severed, and the tribute of respectful memory was due to the last survivors of the men who, in long bygone days, did the work and maintained the honour of the profession.

An important subject which required a few words of notice was the question of examinations. A movement which had been on foot for some little time among members of the Council to deal with this subject in view of the immediate future after the war had during the present year taken a practical form, and the revised syllabus approved by the Council was now in their hands in the current number of the *Journal.*

This had been decided upon by the Council after much careful consideration. Its object was to make it easier for those of their members who had not passed all the examinations necessary for the Fellowship when they went to the war to complete their curriculum on their return with a minimum expenditure of time and labour. He might add that the close attention which had thus been given to the general subject of the qualifying study required for their profession might have a value beyond its immediate use, and that some of the simplifications now adopted might assume permanent form.

Another subject about which a few words were necessary was that of the recent alterations in the Institute's bye-laws, agreed to at a general meeting held at the Institute a week or two previously. The object of these alterations was to give by practical means a properly exercisable freedom of choice to the members in the annual selection of new members of Council, and it was hoped that this power would be freely used.

In conclusion, he would only say one word about the national position in which they found themselves that day. Each year since the war began they had met together thus in the hope that before another such gathering took place it would have ended; and each succeeding year had brought disappointment of that hope. To-day the conflict still raged, and the outlook was as stern as at any time since its outbreak.

None the less, their resolution remained unbroken and their faith unshaken. As a part of a great nation, engaged in a branch of its work which they knew to be important, they would go forward into the future sharing to the full the national spirit, with "courage never to submit or yield" till the objects in view had been attained.

Mr. R. R. Tilt, in seconding the motion, said that on looking at the prosaic details of the accounts the members would see that since the war there had been a reduction in subscriptions by £600, owing mainly to subscriptions having been remitted by the Council in the case of those members taking part in the war. On the other hand they had savings in tutorial and examination fees. With regard to the issue of the *Journal*, he thought the editors ought to be congratulated on the high standard of

that publication during the period when they had no ordinary sessional meetings to record, and he thought that their work was harder than it had ever been. He wished to call special attention to that.

The motion was unanimously agreed to.

Election of Officers.

A ballot was then taken for the election of the President, Vice-Presidents, Council and Officers for the ensuing year; and the Scrutineers subsequently reported that the following Fellows recommended by the Council had been elected:

President.

Geoffrey Marks, O.B.E.

Vice-Presidents.

Ralph Todhunter, M.A.
Arthur Digby Besant, B.A.
Joseph Burn, C.B.E.
James Douglas Watson.

Council.

Samuel John Henry Wallis Allin, C.B.E.
Arthur Digby Besant, B.A.
Joseph Burn, C.B.E.
Charles Ronald Vawdrey Coutts.
William Palin Elderton.
Duncan Cumming Fraser, M.A.
*Lewis Frederick Hovil.
Charles William Kenchington.
Owen Kentish.
Abraham Levine, M.A.
*George James Lidstone, F.R.S.E.
Geoffrey Marks, O.B.E.
*Reginald George Maudling.
Sir George Ernest May, K.B.E.
Henry Edward Melville.
William Peyton Phelps, M.A.
William Charles Sharman.
John Spencer.
Edward Robert Straker.
Alfred Charles Thorne.
Ralph Todhunter, M.A.
*Edward William Townley.
*Harold Moltke Trouncer, M.A.
Samuel George Warner.
Sir Alfred William Watson.
James Douglas Watson.
Arthur Thomas Winter.
Ernest Woods.
William Arthur Workman.
Frank Bertrand Wyatt.

Treasurer.

William Peyton Phelps, M.A.

Honorary Secretaries.

Abraham Levine, M.A.
William Palin Elderton.

* New Members of the Council.

Mr. J. C. Wardrop proposed, and Mr. W. G. Titmuss seconded, the election of Messrs. W. Mouat Jones, E. W. Humphry, and Stanley Hazell as auditors for the ensuing year.

The motion was carried unanimously.

Mr. J. G. Fraser moved that a vote of thanks be accorded the auditors for their work during the past year, and added that he would like to take that opportunity of saying that as one of the distant members of the Institute he greatly appreciated the work done in London by the members in the difficult task of carrying on the work of the Institute.

Mr. Tarn seconded the motion, and it was carried.

Mr. H. J. Baker, in proposing a vote of thanks to the President, Vice-Presidents, the Council, and officers for their services during the past year, said that the Institute was to be congratulated on its good fortune in having Mr. Warner as its President during the past two years. The Vice-Presidents, Council, and officers had done very useful work and had

sacrificed for them some of their leisure from their professional duties. Mr. Jarvis, too, had accomplished his task in a most admirable manner. The affairs of the Institute had in a time of great difficulty been carried on in a manner which added materially to its national usefulness and interest, and the least they could do for those who had borne the heat and burden of the day was to accord them a hearty vote of thanks.

Mr. V. Marr, in seconding the vote of thanks, added his testimony of commendation to the work which had been accomplished, and the motion was carried with applause.

Mr. Warner, in acknowledging the vote on behalf of his colleagues and himself, said that personally he had found great happiness in the service of the Institute during the past two years. To have filled the highest office the profession had to offer was a thing to be marked with a white stone in memory for the years to come. He was very sensible also of all the help he had received during the period from the members of the Council. In vacating the chair, he would just add how gladly he resigned it to so old a friend and so fit a President as Mr. Geoffrey Marks. Mr. Marks had rendered good service to his profession, to the business of life assurance, and to the needs of national service. In his hands they felt sure that the interests of the Institute would be absolutely safe, and in him they would have a worthy representative, of whom they might be proud.

Mr. Geoffrey Marks said he need hardly say how grateful he felt at their electing him to the highest honour which members of the profession could possibly achieve in the world. He had the interests of the Institute at heart, and he hoped the occasion would arise when he would be able to prove to them that whatever he had done in the past was but an earnest of what was to follow for the well-being of the Institute. On occasions like these it was usual for the new President to say how difficult it would be for him to do as much as the ex-President had done, and he thought that no one could say that with more force than he. They had had the benefit of Mr. Warner's guidance for the past two years. Mr. Warner had added a new grace and charm to the Presidential office, and he felt that he could hardly hope to live up to the ideal which he had set. Before he sat down, he hoped that they would forgive him for referring to a personal matter. He found that by an accident, which for him was a very happy one, there were on the Council four Fellows who at one part of their career and for the whole of his, had been his Assistant-Actuaries. Not only that, but one of them was a Vice-President and another was a Honorary Secretary. Those four Fellows could have no illusion as to his qualities and defects, and he knew that he could rely on them for the loyal help which they had always given him in other circumstances. He relied on the help of the Council, because he felt that in these days they must stand together and consider the questions which would come before them in a spirit of good fellowship and solidarity.

The proceedings then terminated.

Additions to the Library.

The following works have been added to the Library since the publication of the *Journal* for October 1917 :

By whom presented (when not purchased).

Actuaries, Faculty of.

Transactions, Vol. VIII. — *The Faculty.*

"Notes on Compound Interest Formulas and Tables", by the late Sir G. F. Hardy, K.C.B. 1918.

By whom presented (when not purchased).

Actuarial Society of America.

Transactions, 1917–18. — *The Society.*

Containing, *inter alia*—

"The War Revenue Act of 3 October 1917, as it affects Life Insurance Companies", by E. E. Rhodes.

"The effect of Glycosuria and of Albuminuria on Mortality", by Dr. Oscar A. Rogers and A. Hunter.

"Reinstatement of Policies", by R. D. Murphy.

"Observations on the methods and publications of the United States Census Bureau", by H. H. Wolfenden.

"Beneficial and Relief Associations: Sickness, Accident and Death Benefits to employees and their Dependents", by A. Hunter.

"Mortality Graphs", by H. Moir.

"Graduation by Symmetrical Coefficients", by J. R. Larus, Jr.

"Joint Mortality Experience of the Aetna Life and Travellers' Insurance Companies on Group Policies", by E. E. Cammack and E. B. Morris.

"Mortality by order of Birth", by P. H. Evans.

"Determination of Makeham Graduation Constants by means of Equivalent Ages", by P. C. H. Papps.

"An Investigation of the mortality prevailing among the American Clergy in its relation to other Classes of the Population and its bearing on a new Standard Table of Mortality", by W. S. Nichols.

"Note on Double Indemnity Clauses in Life Insurance Contracts", by W. A. Hutcheson.

Actuarial Society of New South Wales.

Transactions, 1918. — *The Society.*

American Mathematical Society.

Transactions, 1917–18. — *The Society.*

American Statistical Association.

Transactions, 1917–18. — *The Association.*

Argentine Republic.

Boletin de la Direccion General de Estadistica y Departamento Provincial del Trabajo. 8vo. 1918. — *The Department.*

Ball (W. W. Rouse).

A short account of the History of Mathematics. 6th edit. 8vo. 1915. — *Purchased.*

Cantelli (F. P.).

Sull' aumento di mortalità dovuto alla guerra. Riflessioni critische di metodologia statistica. Rome. 1917. — *The Author.*

Casualty Actuarial and Statistical Society of America.

Proceedings, 1917–18. — *The Society.*

	By whom presented (when not purchased).
Central Control Board (Liquor Traffic).	
Alcohol. Its action on the Human System. 8vo. 1918.	*Purchased.*
Charlier (C. V. L.).	
Über das Fehlergesetz. 1905. Die zweite Form des Fehlergesetzes. 1905. Über die Darstellung willkürlicher Funktionen. 1905. Weiters über das Fehlergesetz. 1908. Die Strenge Form des Bernoulli'schen Theorems. 1909. Contributions to the Mathematical Theory of Statistics. 5. Frequency Curves of type A in Heterograde Statistics. 6. The Correlation Function of type A. 7. Frequency Curves of Compound Functions. 8. On Multiple Correlation. 1914–15. The Radiation Law applied to stellar photometry. 1915. Studies in Stellar Statistics. 3 Parts. 4to. 1912–16.	*Dr. A. Lindstedt.*
Chartered Insurance Institute, Journal of the	
Vol. XX. 8vo. 1917.	*The Institute.*
De Morgan (A.).	
A Budget of Paradoxes. Second Edition by D. Eugene Smith. 2 vols. 8vo. 1915.	*Purchased.*
Denmark.	
Beretning fra Forsikringsraadet, 1917. Statsanstalten for Livsforsikrings. Den 9de femaarige Risiko-Opgorelse. Copenhagen. 1917.	*The Danish Government.*
Downing (E. R.).	
The Third and Fourth Generation. An Introduction to Heredity. 8vo. Chicago. 1918.	*Purchased.*
Durell (Col. A. J. V.).	
The Principles and Practice of the System of Control over Parliamentary Grants. 8vo. 1917.	*Purchased.*
Economic Society (Royal).	
Journal of the, 1917–18.	*Purchased.*
Fisher (A.).	
Note on the Construction of Mortality Tables by means of Compound Frequency Curves. Boston, Mass. 1917.	*Dr. F. L. Hoffman.*
Hoffman (Dr. F. L.).	
The Malaria Problem in Peace and War. Newark, N.J. 1918.	*The Author.*
Holland.	
Archief voor de Verzekerings-Wetenschap. 1917–18.	*The Society.*
Nationale Levensverzekering-Bank. Verslag over het jaar 1917.	*The Company.*
Huygens (Christiaan).	
Du Calcul dans les Jeux de Hasard. Rédigé par D. J. Korteweg. Extrait des Oeuvres complètes. Tome XIV. 4to. N.D.	*D. J. Korteweg.*

	By whom presented (when not purchased).
Institute of Bankers.	
Journal of the, 1917–18.	*The Institute.*
Insurance Institute of New South Wales.	
Transactions, 1916.	*The Institute.*
Insurance Institute of Toronto.	
Proceedings, 1917–18.	*The Institute.*
Joffe (S. A.).	
Interpolation Formulæ and Central Difference Notation. 1917.	*The Author.*
Knibbs (G. H.).	
Australian Joint Life Tables, 1901–1910. 8vo. Melbourne. 1917.	*The Author.*
Life Offices' Association.	
Minutes, etc., of the Meetings of the Standing Committee, 1917–18.	*The Association.*
London Mathematical Society.	
Proceedings, 1917–18.	*The Society.*
National Health Insurance.	
Report on the Administration of National Health Insurance during the years 1914–17. 8vo. 1917.	*The Government Actuary.*
Reports of the Medical Research Committee, 1915–18. 8vo.	*Purchased.*
Summary of the Principal Provisions of the Act of 1918.	*The Government Actuary.*
Nicholson (J. S.).	
War Finance. 8vo. 1917.	*Purchased.*
Parliamentary Papers.	
Assurance Companies. Returns to the Board of Trade. 1917.	*The Board of Trade.*
Censuses.	
1911. England and Wales. Vol. XIII. Fertility of Marriage. Part I. Fol. 1917.	*The Registrar-General.*
Colonies.	
Canada.	
Report of the Superintendent of Insurance for the year 1916. Insurance Companies. Abstract of Statements for the year 1917.	*The Government Insurance Dept.*
New South Wales.	
Friendly Societies, &c. Report of the Registrar for 1917. Official Year Book, 1917. Statistical Register for 1916–17 and previous years. Vital Statistics. Report for 1917 and previous years.	*The Government of N.S.W.*

By whom presented (when not purchased).

Parliamentary Papers—*continued.*

Colonies—*continued.*

New Zealand.

Friendly Societies. Forty-first Annual Report of the Registrar, 1917.
Official Year Book, 1917.
Statistics of the Dominion for the years 1916–17.
Results of a Census of the Dominion, 1916. Part I—Population. Appendices A–D. Fol. Wellington. 1918. — *The Government of N.Z.*

Victoria.

Friendly Societies. Thirty-ninth Annual Report of the Government Statist, 1916.
Twenty-seventh Annual Report of the Registrar, 1916. — *The Government of Victoria.*

Western Australia.

Friendly Societies. Report of Proceedings by the Registrar for the year ended 30 June 1917.
Statistical Register for 1916 and previous years. — *The Government of W.A.*

India.

Life Assurance Companies. Returns of Companies doing business in British India, 1916. Fol. Simla. 1917. — *H. G. W. Meikle.*

Registrar-General. England.

Seventy-ninth Annual Report of Births, Deaths, and Marriages, 1916. — *The Registrar-General.*

Pearson (James).

The Elements of the Calculus of Finite Differences, created on the method of Separation of Symbols. 8vo. Cambridge. 1849. — *Purchased.*

Periodicals.

Accountants' Magazine.
American Bankers' Magazine. — *Purchased.*

American Economic Review. — *The Editor.*

Bankers' Magazine.
Economist. — *Purchased.*

Insurance Record.
Journal of Political Economy. (Chicago).
Post Magazine.
Post Magazine Almanack. — *The Editor.*

The "Secretary." — *Institute of Secretaries.*

Ross (Lt.-Col. Sir Ronald).

An application of the Theory of Probabilities to the study of Pathometry. 8vo. 1916. — *Purchased.*

Royal Astronomical Society.

Memoirs of the. Vol. LIII, Part III. 4to. 1918. — *The Society.*

Scandinavia.

Skandinavisk Aktuarietidskrift, 1918. — *The Editor.*

Schooten (Franz van).

Exercitationum Mathematicarum. 4to. 1657. — *Purchased.*

	By whom presented (when not purchased).
"Scientific Monthly."	
New York. April, 1917.	*Purchased.*
Containing, *inter alia*—	
"Thrift from the Life Insurance view-point", by E. E. Rittenhouse.	
"Some fallacies of Compulsory Health Insurance", by Dr. F. L. Hoffman.	
"Insurance as a basis of Social Economy", by H. Fiske.	
"Recent developments in the Life Insurance field", by R. L. Cox.	
"Life Insurance and Life Conservation", by E. L. Fisk.	
"Life Insurance and the War", by S. S. Huebner.	
Secrist (H.).	
An Introduction to Statistical Methods. 8vo. New York. 1917.	*Purchased.*
Sellar (A. S.).	
Tables for computing Income Tax at 6/– in the £, showing Income Tax and also net amount at one inspection. 8vo. 1918.	*The Author.*
Spain.	
Conferencia de Seguros Sociales, 24–31 October 1917. 8vo. Madrid. 1917.	*F. Lopez-Valencia.*
Statistical Society (Royal).	
Journal of the, 1917–18.	*The Society.*
Steffensen (Dr. J. F.).	
On certain inequalities between mean values, and their application to actuarial problems. 1918.	*The Author.*
Stirling (James).	
The Differential Method; or a Treatise concerning Summation and Interpolation of Infinite Series. Translated by Francis Holliday. 4to. 1749–64.	*Purchased.*
Sweden.	
Enskilda Försäkringsanstalter ar 1916, av Kungl. Försäkringsinspektionen.	*The Swedish Government.*
Utjämning av sjutton Svenska Livförsäkringsbolags Dödlighetstabeller för Lakareundersökta Livförsäkrade. By G. Stoltz. 8vo. Stockholm. 1917.	*The Author.*
Switzerland.	
Rapport du Brueau Fédéral des Assurances sur les Entreprises privées en matière d' Assurances en Suisse. 1915.	*The Swiss Government.*
United States of America.	
Official Publications. First Annual Report on Birth Statistics for the Registration Area of the United States, 1915. Annual Report of the Bureau of the Census on Mortality Statistics, 1915.	*U.S Census Bureau.*
Connecticut.	
Fifty-third Annual Report of the Insurance Commissioner, 1918.	*The Commissioner.*

	By whom presented (when not purchased).
United States of America—*continued.*	
Official Publications—*continued.*	
Illinois.	
Report of Pension Laws Commission. 8vo. 1916.	*D. F. Campbell.*
Massachusetts.	
Sixty-second Annual Report of the Insurance Commissioner (Life and Miscellaneous), 1916.	*The Commissioner.*
New Jersey.	
Reports of the Pension and Retirement Fund Commission. Newark, N.J. 1917–18.	*Dr. F. L. Hoffman.*
New York.	
Report on the Pension Funds of the City of New York. Part III. 8vo. 1918.	*George B. Buck.*
Proceedings of the Conference on Social Insurance, 5–9 December 1916. 8vo. Washington. 1917.	*U.S. Dept. of Labour.*
Wicksell (S. D.).	
Some Theorems in the Theory of Probability, with special reference to their importance in the Theory of Homograde Correlation. 1916.	*Dr. A. Lindstedt.*
On Logarithmic Correlation, with an application to the distribution of ages at first marriage. 1917.	*Dr. A. Lindstedt.*
On the Genetic Theory of Frequency. 1917.	*Dr. A. Lindstedt.*
On the Correlation of Acting Probabilities. 1918.	*Dr. A. Lindstedt.*
The Correlation Function of type A, and the Regression of its characteristics. 4to. 1917.	*Dr. A. Lindstedt.*
Withers (H.).	
The Business of Finance. 8vo. 1918.	*Purchased.*

NEW BYE-LAWS.

At a Special General Meeting of the Fellows and Associates of the Institute of Actuaries, held on 23 May 1918, it was moved that certain alterations be made in the Bye-Laws. The motion was adopted by the Meeting, and the resolutions were duly confirmed at a further Special General Meeting, held on 17 June 1918.

The proposed amendments, as follows, have now received the sanction of the Privy Council :

AT THE COUNCIL CHAMBER, WHITEHALL,

THE 28 JUNE 1918.

By the Lords of His Majesty's Most Honourable Privy Council.

WHEREAS the Institute of Actuaries, in exercise of the powers in that behalf conferred on it by the Charter of the said Institute, did, by Resolution of a General Meeting of the Fellows and Asso-

ciates of the said Institute, duly confirmed at a subsequent General Meeting of the said Fellows and Associates, amend certain of the Bye-Laws of the said Institute, and did rescind certain others of the said Bye-Laws and make others in their stead :

And whereas the said amended and substituted Bye-Laws have in compliance with Article 45 of the Charter been submitted to the Lords of the Council :

Now, therefore, Their Lordships, having taken the said Bye-laws (a copy whereof is hereunto annexed) into consideration, are pleased to allow the same.

Almeric FitzRoy.

Bye-Laws referred to in the foregoing Order.

At a Special General Meeting of the Fellows and Associates of the Institute of Actuaries, duly convened and held at Staple Inn Hall, on Thursday, the 23 May 1918, and at a subsequent Special General Meeting of the said Fellows and Associates, also duly convened and held at the same place, on Monday, the 17 June 1918, the following portions of Bye-Laws 30, 32, 34, and 35 were rescinded, namely :

30. The portion comprising the words "*shall be elected* "*every year at the Annual General Meeting from the Fellows of* "*the Institute. He*"

32. The portion comprising the words "*and shall be elected* "*every year at the Annual General Meeting from the Fellows of* "*the Institute.*"

34. The portion comprising the words "*shall be elected* "*every year at the Annual General Meeting from the Fellows of* "*the Institute. He*"

35. The portion comprising the words "*shall be elected* "*every year at the Annual General Meeting from the Fellows of* "*the Institute. They*";

and Bye-Laws 4, 5 and 73 were also rescinded, and in substitution thereof the following Bye-Laws, to be numbered respectively 4, 5 and 73, were duly made, passed and confirmed, namely :

"4. (*a*) The President, the four Vice-Presidents, the "Treasurer and the Honorary Secretaries shall go out of office "as such at the close of the Annual General Meeting of the "year for which they were elected, and, except as provided "by Bye-Laws 30, 32, 34 and 35, shall be eligible for re-election.

"(*b*) The Council shall, at a Council Meeting to be held "not less than fifty days before the Annual General Meeting "in each year, elect from the then Members of the Council "the President, the four Vice-Presidents, the Treasurer and "the Honorary Secretaries for the ensuing year. Such election "shall be conducted in such manner as the Council may, from "time to time, prescribe."

"5. (A) At every Annual General Meeting five Members of "the Council (which number shall include any ceasing to hold "office as Members of the Council under Bye-Law 7) shall retire "and five Fellows shall be elected in manner hereinafter "provided to fill the vacancies so arising. The Members so "to retire (apart from any ceasing to hold office as Members "of the Council under Bye-Law 7) shall be chosen by the "Council at a Meeting to be held not less than fifty days before "such Annual General Meeting, and the Members so chosen "shall not be eligible for re-election before the Annual General "Meeting next following that at which they go out of office, "but any Members ceasing to hold office as Members of the "Council under Bye-Law 7 shall be eligible for re-election.

"(B) The Council shall, not less than forty days before each "Annual General Meeting, send to each Fellow and Associate, "at his usual or last known address, a list containing the "names of those Members of the Council who will retire under "Section A of this Bye-Law and under Bye-Law 7, distinguishing "those eligible and those not eligible for re-election.

"(C) The Fellows and Associates shall be entitled to nomi-"nate any Fellow or Fellows for election to the vacancies in "the Council arising by retirement under Section A of this "Bye-Law, or under Bye-Law 7 or otherwise, but no nomination "shall be valid unless it be signed by not less than seven "Fellows or Associates (of whom not fewer than four shall be "Fellows) and lodged at the Offices of the Institute not less "than twenty-one days before the date of the Annual General "Meeting.

"(D) If sufficient nominations to fill up all the vacancies "in the Council, arising as aforesaid, are not so lodged the "Council shall nominate a Fellow or Fellows to fill up any "remaining vacancies.

"(E) The Council shall, not less than seven days before "each Annual General Meeting, send to each Fellow and "Associate, at his usual or last known address :

"(*a*) A list containing the names of those Fellows elected "under Bye-Law 4 to the offices above mentioned,

"and

"(*b*) A list containing the names of the Fellows duly "nominated as aforesaid to fill the vacancies in the Council "arising as aforesaid.

"(F) If the names in the last-mentioned list shall exceed "the number of vacancies as aforesaid such list shall be the "balloting list for the election at the Annual General Meeting "of Fellows to fill the vacancies in the Council, and at such "Meeting each Fellow and Associate may vote for as many "candidates as there are vacancies to be filled, and, if he shall "vote for more candidates than there are vacancies to be "filled, his balloting paper shall be rejected by the Scrutineers.

"(G) The Scrutineers shall be two or more in number to be "chosen previous to the balloting for election from those "present by those present at the Meeting, and they shall "receive the votes and report the result of their scrutiny to "the Chairman before the close of the Meeting."

"73. At the Annual General Meeting there shall be elected "in manner provided by the Bye-Laws the Members to fill the "vacancies in the Council arising under section A of Bye-Law 5 "or under Bye-Law 7 or otherwise, and the Auditors for the "ensuing year."

The Common Seal of the Institute of Actuaries was hereunto affixed by order of the Council this 17 June 1918, in the presence of

L. S.

Geoffrey Marks, *President.*

A. Levine, *Joint Hon. Secretary.*

Members of the Council.

COURSE OF READING

recommended by the Board of Examiners for the guidance of Students in connection with the revised Examination Syllabus (see p. 72 of the present volume of the *Journal*).

The latest editions of Text-Books are referred to in each case. The references to Transactions of Actuarial Societies have been restricted with the object of directing special attention to particular phases of the subjects. The Board accept no responsibility for any opinions expressed in the papers recommended or in the discussions following the papers, which should also be studied.

Questions in Parts III *and* IV *will be as far as possible of a practical nature, involving the application of the principles discussed in the suggested course of reading, and candidates should also be familiar with current actuarial and insurance topics.*

PART I.

Section A.

H. S. Hall and S. R. Knight: "Higher Algebra." (Macmillan & Co.).

No questions will be set on Interest and Annuities, Advanced Convergency and Divergency of Series, Continued Fractions, Indeterminate Equations of the Second Degree, Theory of Numbers, Inverse Probabilities, Determinants, Elimination, or Cubic and Biquadratic Equations.

J. Burn and E. H. Brown: "Elements of Finite Differences", Part I. (C. & E. Layton).

Institute *Text-Book,* Part II, chapters xxii, xxiii, and xxiv, sections 1–20.

J. Edwards: "Differential Calculus for Beginners." (Macmillan & Co.).

Chapters 1–7 *and* 13, *omitting Trigonometrical references.*

J. Edwards: "Integral Calculus for Beginners." (Macmillan and Co.).

Chapters 1–6 *and the general propositions in Chapter* 8, *omitting Trigonometrical references.*

Institute *Text-Book,* Part I, chapter ix.

J.I.A., vol. xl, p. 116, section 1 (W. P. Elderton); vol. xliv, p. 402, sections 1–5 (G. J. Lidstone and S. E. Macnaghten).

Section B.

Institute *Text-Book,* Part I (excluding chapters ix and x).

PART II.

Institute *Text-Book,* Part II (excluding chapters xix, xx, xxii, xxiii, and xxiv, sections 1–20).

J.I.A., vol. xxii, p. 407 (T. B. Sprague); vol. xl, p. 302 (Actuarial Note, No. 3); vol. xli, p. 97 (G. J. Lidstone); vol. xliii, p. 99 (W. P. Elderton); vol. xliv, p. 402 (G. J. Lidstone and S. E. Macnaghten).

Transactions of the Faculty of Actuaries, vol. v, p. 130 (W. Borland).

PART III.

Section A.

W. P. Elderton and R. C. Fippard: "The Construction of Mortality and Sickness Tables." (A. & C. Black).

Supplement to the 75th Annual Report of the Registrar-General, Part I, Life Tables (Cd. 7512, 1914). (The review of this Report, *J.I.A.*, vol. xlix, p. 96, might conveniently be read as an introduction to it.)

Sir George F. Hardy: "The Theory of the Construction of Tables of Mortality, &c.", chapters i and ii. (C. & E. Layton).

J.I.A., vol. xxi, p. 406 (T. B. Sprague); vol. xxvi, p. 77 (T. B. Sprague); vol. xxxii, p. 371 (Sir G. F. Hardy); vol. xxxviii, p. 11 (G. J. Lidstone); vol. xxxviii, p. 501 (Sir G. F. Hardy); vol. xli, p. 348 (G. J. Lidstone); vol. xlvii, p. 548 (Extract from Cd. 6907, 1913).

Journal of the Students' Society, vol. i, No. 3, p. 43. (Editorial); vol. i, No. 2, p. 44 (W. P. Elderton).

Section B.

Students should, if possible, study some system of Valuation in actual use, and should possess a general knowledge of current financial conditions.

J.I.A., vol. xxxvii, p. 57* (S. G. Warner); vol. xxxvii, p. 453* (G. King); vol. xxxviii, p. 1 (G. J. Lidstone); vol. xxxviii, p. 69* (Sir G. H. Ryan); vol. xxxviii, p. 385* (D. C. Fraser); vol. xl, p. 122 (D. C. Fraser); vol. xli, p. 18* (J. Buchanan); vol. xlii, p. 145 (G. King); vol. xlii, p. 161 (C. R. V. Coutts); vol. xlii, p. 409 (G. J. Lidstone); vol. xliv, p. 261 (G. J. Lidstone); vol. xlviii, p. 1 (W. P. Elderton); vol. xlviii, p. 121 (A. E. King); vol. l, p. 231 (P. H. McCormack).

Transactions of the Faculty of Actuaries, vol. vi, p. 93 (A. Fraser).

Journal of the Students' Society, Special Number on "Valuation of Liabilities and Distribution of Surplus", vol. i, No. 2, p. 24 (E. A. Woodall) and p. 31 (R. C. Fippard); vol. i, No. 4, p. 5 (R. C. Simmonds).

Schedules IV and V of the Assurance Companies Act, 1909, and the Returns relating to Life and Employers' Liability business to the Board of Trade under that Act.

* These papers, in conjunction with Returns to the Board of Trade, should be studied in the first instance.

PART IV.

Section A.

Students should possess a general knowledge of current financial conditions, especially as affecting Trustee and gilt-edged securities.

(1) General provisions of the Acts relating to Life Assurance Companies, &c. :

Assurance Companies Act, 1909.

J.I.A., vol. xliv, p. 462 ; vol. xlv, p. 257 (A. R. Barrand).

Friendly Societies Acts, 1896 and 1908, and Collecting Societies and Industrial Assurance Companies Act, 1896 (actuarial and financial aspects only).

Legal Notes in *J.I.A.*, on the above Acts.

Bunyon on the Law of Life Assurance (Fifth Edition), chapters vi–x. (C. & E. Layton).

(2) Life Interests and Reversions, &c. :

Institute *Text-Book*, Part II, chapter xix.

J.I.A., vol. xxvii, p. 107 (T. B. Sprague) ; vol. xxxvi, p. 81 (correspondence with Inland Revenue) ; vol. xl, p. 317 (C. R. V. Coutts) ; vol. xlvi, p. 406 (G. J. Lidstone).

Transactions of the Faculty of Actuaries, vol. i, p. 79 (N. Campbell) ; vol. iii, p. 201 (A. E. Sprague).

Journal of the Students' Society, vol. i, No. 4, p. 50 (Editorial) and No. 5, p. 24 (Editorial).

Section B.

Students should, if possible, study the practice of Offices with regard to the subjects in this section, including some system of Life Assurance Book-keeping in actual use, and forms of Returns to the Board of Trade.

(1) Law relating to Life Assurance Contracts :

Bunyon on the Law of Life Assurance (Fifth Edition), chapters i to v, and chapter xi to end. (C. & E. Layton).

J.I.A, vol. xxxiii, p. 205 (A. R. Barrand); vol. xxxiii, p. 373 (T. B. Sprague); vol. xxxv, p. 371 (G. J. Lidstone); vol. xli, p. 109 (A. R. Barrand).

Legal Notes in *J.I.A.*, vol. xli, p. 409, *et seq.*

Transactions of the Faculty of Actuaries, vol. ii, p. 343 (A. E. Sprague).

(2) Life Assurance Accounts, &c.:

A. E. Sprague: "Insurance Companies' Accounts." (C. & E. Layton).

Journal of the Students' Society, vol. i, No. 1, p. 36 (S. E. Macnaghten); vol. i, No. 5, p. 22 (Editorial).

Income Tax Act, 1918:

Sections 32, 33 and 46.

Schedule D, Cases I and II, Rule 15; Case III, Rule 3; Case IV, Rules 1 and 2; Case V, Rules 1, 2 and 3.

General Rules applicable to Schedules A, B, C, D and E: Nos. 19 and 21.

(3) Office Premiums, &c.:

Transactions of the Faculty of Actuaries, vol. ii, p. 207 (H. Moir).

Insurance Guide and Handbook, vol. i, chapters v and vi. (C. & E. Layton).

(4) Extra Premiums:

Journal of the Students' Society, Special Number on "Extra Risks."

Section C.

Students are expected to be familiar with the principal provisions of the National Health Insurance Acts.

(1) Friendly Societies, &c.:

J.I.A., vol. xxvii, p. 245 (Sir G. F. Hardy).

Sir A. W. Watson: "Lectures on Friendly Societies." (C. & E. Layton).

Sir A. W. Watson: "Account of an Investigation of the Sickness and Mortality Experience of the I.O.O.F. Manchester Unity, 1893–1897." (C. & E. Layton).

Journal of the Students' Society, vol. i, No. 5, p. 7 (R. C. Simmonds), and p. 18 (Editorial).

(2) National Health Insurance:

Actuaries' Report on the Schemes embodied in the National Insurance Bill, 1911 (*J.I.A.*, vol. xlv, p. 406).

*Reports on the Administration of National Health Insurance (1912–13, Cd. 6907, pp. 16–35 and 552–601; 1913–14, Cd. 7496, pp. 29–64; 1914–17, Cd. 8890, pp. 7–15). Review, *J.I.A.*, vol. xlviii, p. 109.

*Reports of the Departmental Committee on Approved Society Finance and Administration (Cd. 8251, Cd. 8396, Cd. 8451). Reviews, *J.I.A.*, vol. l, pp. 99 and 221.

(3) Statistics:

W. P. and E. M. Elderton: "Primer of Statistics." (A. & C. Black).

*Supplement to the 65th Annual Report of the Registrar-General. Review, *J.I.A.*, vol. xliii, p. 230.

Report on the Administration of National Health Insurance, 1912–13, Cd. 6907, pp. 552–601.

Transactions of the Faculty of Actuaries, vol. v, p. 1 (J. C. Dunlop); vol. vii, p. 357 (J. C. Dunlop and R. M. Hunter).

* The Reviews of these Reports might conveniently be read as introductions to them.

Obituary.

FREDERICK GEORGE GOODYEAR, M.C., Probationer of the Institute, 2nd Lieutenant, London Regiment.

Died of Wounds 9 *December* 1917.

EDWARD MADDISON DOVE, M.C., Probationer of the Institute, Lieutenant, 8th Battalion East Surrey Regiment.

Killed in Action 23 *March* 1918.

Hugh Thomas Kay Robinson, D.S.O. with Bar, Fellow of the Institute, Lieut.-Colonel, Royal Sussex Regiment.

Killed in Action 26 *April* 1918.

Alan Davidson Steed, Probationer of the Institute, Private, 15th London Rifles (Queen's Westminsters).

Killed in Action 30 *April* 1918.

George Leslie Lewis Carter, Student of the Institute, Naval Instructor, Royal Naval College, Dartmouth.

Died on Service 29 *July* 1918.

Edwin Charles Kaye Clarke, M.C., Probationer of the Institute, Captain, London Regiment.

Killed in Action 31 *August* 1918.

Henry Batten Keable, Associate of the Institute, Lieut., R.N.V.R., H.M.S. "Manxman."

Died on Service 11 *September* 1918.

Arthur Skelton Gregory, Probationer of the Institute, 2nd Lieutenant, 6th Battalion Loyal North Lancashire Regiment (attached 2nd Batt. East Lancs.).

Killed in Action 21 *September* 1918.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Opening Address by the President, Geoffrey Marks, Esq., O.B.E.

[Delivered 16 December 1918.]

Gentlemen,—It is customary on these occasions to refer briefly to those Fellows of the Institute who have died since the last Presidential address. In the ordinary course of nature there have gone Mr. Chas. Stevens and Mr. J. G. Priestley, the last surviving members of the Actuaries' Club who became Fellows under the Charter. They had attained great ages—one rather less and one rather more than 90—and their names recall little to us of this generation except a tradition of courtliness and general culture which were attractive features of the older generation to which they belonged. Two other Fellows, Mr. James Graham and Mr. W. F. Somerville, spent their professional lives largely outside our immediate circle, the one in Australia and the other in the North of England. They were therefore little known to us in London, but they were both loyal members of our body so far as their opportunities allowed them to give proof of it. Mr. H. W. Andras and Mr. Geo. Todd, on the other hand, were closely identified with us and with our work throughout their careers. They had both filled almost every one of our high offices, and their careers and characters showed much in common, especially a personal charm and genius for

friendship, which will ensure their being held in continued and affectionate remembrance. A few weeks ago all who knew him here and at the Actuaries' Club were saddened by the announcement of Vyvyan Marr's sudden death. He too was one of our most loyal sons, always ready to devote to the service of the Institute abilities which had gained for him a considerable reputation in connection with pension-fund work, that difficult branch of our activities which chiefly appealed to him.

In our War record we may properly feel and express a pride mingled with regret. According to the latest returns 424 of our members, a very large proportion of our total number, have enlisted or been mobilized. Of these 51 have been killed in action, 10 have died of wounds, and 6 from other causes while on active service. With all others who have served and died for their country they will always be held in grateful remembrance. Honour and Peace to them!

Many of our members have won distinctions in the War, but I think that without being invidious I may mention especially two, both Fellows—one a soldier and the other a sailor—H. T. Kay Robinson, who unhappily is dead, and W. R. Ashton, whom, having regard to the nature of his services, we may well congratulate on being alive. Robinson at the time of his death had attained the rank of Lieut.-Colonel. He won his first D.S.O. for gallantry at the Schwaben Redoubt, a bar to it for "fine leadership and courage" at "Tower Hamlets," while a second bar was awarded to him after his death. His Brigadier wrote of him that he was a "born leader of men and quite fearless in action," while one of his officers describes him as "the bravest man I have ever met." If there is any consolation for the death of a man so gallant and so honoured it lies in such tributes as these.

We have read recently something of the work of the "Q" boats and of the cool resource, the calm and sustained courage of those who manned them. There is no more noble record of bravery or endurance in the annals even of the Navy. Ashton is one of those who have earned undying fame in this perilous work. He was a member of the R.N.V.R. before the War and was mobilized in 1914, but did not begin his "Q" ship career until 1917. In the course of it he earned the D.S.C. with two bars and the D.S.O., and was promoted

out of course to Staff-Paymaster for services in action. His last action was on the 24 December 1917, when his ship went down after sinking its opponent.

Another Fellow who has earned high distinction is Lieut.-Colonel H. J. Wenyon, who has won the D.S.O. with a bar. Yet another Fellow, Capt. E. H. Lever, on service with the British Forces in Italy, has received the following French decorations: The Legion of Honour, and the Croix de Guerre twice—"with palm" and "with gold star"—the first and second orders. An Associate, T. P. Wansbrough, was also a Lieut.-Colonel when I saw him last. He rapidly attained his rank in the comparatively peaceful but arduous and responsible work of the R.A.S.C., in which his powers of organization were quickly recognized and rewarded.

These are a few outstanding examples of the honours gained by our men, but our records are very incomplete and I earnestly beg that all who have any knowledge of further distinctions will bring them to the notice of the Council. It is due to the Institute, but still more to the men, that this should be done.

Much has been written and much said in the attempt to reconcile what an actuary is to-day with the historical origin of the distinctive name under which he is classified among the professions, and to trace the steps by which the shorthand writer of B.C. 40 developed into the G. F. Hardy, the Lidstone, or the Elderton of our time.

With this interesting but somewhat futile speculation I do not propose to concern myself much to-night.

Nevertheless a definition is necessary to my purpose to-night because I wish:

(A) To find a comprehensive description of the actuary as he might be under present conditions, and would be if he were to attain the ideal which few or none can reach.

(B) To see how far the modified training which the Institute now proposes to give to its students will fit them to become, not of course the ideal actuary, but the sound and reliable one, capable of dealing with all the ordinary problems which are likely to come before him.

(C) To indicate some directions in which the usefulness of our members may be enlarged for the advantage

of the State and the benefit of the Institute and of themselves; and

(D) To suggest methods by which students may obtain the necessary training and experience for this further work, while still remaining members of the Institute and still regarding it as their *Alma Mater*.

Only thus do I conceive that in these days, and in those which are to come, can the Institute maintain its right to the high place which it has now reached, and fulfil its duties towards its Members and the Commonwealth. It is my belief, and indeed my hope, that one result of the War will be that there will be far less room for the half-hearted or the wholly ineffectual in the future than there has been in the past, whether they be institutions or individuals. I guard myself against misconception by saying that in my view at no previous time has the reputation of the Institute and of the actuary stood higher than it does now, largely because of the conspicuous ability of those members of our body who in the past few years have been associated in their professional capacity with the Government and Government Departments.

(A) What is an Actuary? The best dictionary definition which I know is in Murray's Oxford Dictionary, quoted in the *Journal* (vol. xxvi, p. 389):

"An official in an insurance office, whose duty it is to "compile statistical tables of mortality, and estimate "therefrom the necessary rates of premium, &c., or one whose "profession it is to solve for insurance companies or the "public, all monetary questions that involve a consideration "of the separate or combined effect of interest and "probability in connection with the duration of human life, "the average proportion of losses due to fire or other "accidents, &c.

My own definition would be rather more general:

"One whose profession it is to devise means to solve all "questions involving the application of the theory of "probability to human affairs, whether in conjunction with "the rate of interest or not, and to apply them to the solution "of practical problems."

This definition is intentionally wide, and the ideal which it presents is probably impossible of attainment by any human being. It implies a depth and breadth of knowledge to acquire which would leave no time or room for the practical

experience of men and affairs, without which the practising actuary at least is little or nothing worth.

As we have lately been reminded, a former President said: "An actuary should be a man of general culture, with a "knowledge both of books and men, and the more he has of "both the better." In this I heartily concur, but in the same paper the President committed himself to the definition of an actuary as a "scientific financier" and in this, as things were then and are at present, I can hardly follow him. The description might serve to satisfy the lukewarm interest of one's neighbour at dinner, and has, in fact, been useful in that connection, but as a practical definition it seems to me to exaggerate in one direction the possibilities of our training, while limiting them in others.

Before I leave this point perhaps I may make one other quotation from an extract printed in the *Journal* (vol. xxx, p. 344.) It may interest you, as it did me, because I think that it shows a wise prevision of the possibilities of actuarial training, and of the true character of the Institute, all the more remarkable because it was written so long ago as 1848, on the occasion of the Institute's foundation, and by a man who was not an actuary. He speaks of:

> "The science of which an actuary practises the "application, and which, though at present it principally "relates to life contingencies, yet must be held to include "all contingencies to which calculation of probability can "be applied."

Of the Institute, he says:

> "Such a society should be as republican in its "constitution and as liberal in its principles as those "which have preceded it. No man should have any rank "except what he makes for himself out of the opinion of "his comrades; no class of privileged members should "exist."

The Institute has endeavoured during its 70 years of life faithfully to fulfil the conditions of its legitimate existence outlined here by John Francis. So I believe, and it is in that belief that I find the greatest comfort and satisfaction in occupying this chair and addressing you to-night.

(B) As you know under the pressure of the war the Institute has lately revised the syllabus of its examinations. The object of this revision was to modify the examination

standard to such an extent as would relieve those members and students who have spent the last four years in the arduous service of their country from the extended course of study necessary to success in pre-war examinations, while still maintaining a sufficient standard of general and actuarial knowledge to qualify them to act in all the capacities which an actuary is ordinarily called upon to fulfil. This modification is at present to be considered as an emergency measure, but my suggestions to-night are intended to direct your attention to the possibility that the present arrangement may be continued and extended, because I believe that it is along the lines proposed by the Examination Committee and adopted by the Council that the Institute is capable of attaining its widest and fullest development. I may add that this aspect of the case was fully in the minds of the Committee and the Sub-Committee which at the beginning of the year exhaustively examined the subject.

The modified syllabus which is now in your hands has been supplemented by a course of reading drawn up by the Board of Examiners which seems to me to meet admirably all the necessities of students within the limits laid down in the syllabus. In the past as you know unless the student were fortunate enough to have the benefit of the advice of experienced tutors or private coaches (a condition which in itself presents some dangers) his course of reading embraced numbers of papers, not all of equal merit, many of which duplicated his work to a great extent. In the courses of reading now prescribed for each section of the examination he will find all the equipment which is necessary to the ordinary actuary, and sufficient leisure to enable him to explore those byways in which he feels himself to be specially interested.

Certain branches of our work which I think might quite reasonably be made separate subjects of further development are still included in the revised syllabus. Nevertheless, inasmuch as Sections B and C of Part IV are alternative, and as I conceive that the majority of candidates will select Section B in preference to Section C, because the practical experience of most of them will lie in the subjects comprised in Section B, the effect will be that Section C and the subjects of which it treats will become purely voluntary.

I suggest therefore that Section C should be taken out of

the syllabus, and the subjects included in it (and possibly others) be dealt with on lines which I will describe hereafter. Sections A and B might be combined into one final section, and possibly three papers set in it, if it is thought desirable to include in it some questions on law, and on the elements of statistics in which a preliminary course of reading could easily be prescribed. Personally I think that it would not be desirable be exclude from the final examination all questions of law, because many of our every day experiences involve some knowledge of the law relating to life assurance companies and contracts, and of that governing mortgage transactions.

You will observe that I am not reducing the compulsory requirements for the final examination in the case of ordinary students more than they have been reduced by the Council. I think that the syllabus so compressed would provide adequate training for the ordinary actuary, and I am now considering the matter only with reference to him and to his needs.

You will however gather that although I feel that the reduced syllabus provides sufficient training for that class of man, the omission of Section C would prevent it meeting the needs of the young actuary who wishes to specialize in certain branches of our work, while all questions relating to Finance are omitted both from the revised syllabus and from my suggested modifications of it. To this aspect of the question I propose to refer later, and to offer some suggestions as to the mode in which the omission may be remedied on voluntary lines.

(c) In what directions useful to itself and others may the Institute hope to increase or develop its activities? These include in addition to certain aspects and varieties of insurance:

(*a*) The great question of National Insurance, whether cf health or against unemployment, and of Old Age Pensions.

(*b*) Better and more complete treatment, on actuarial as distinguished from statistical lines, of the Census results.

(*c*) Public Health and the investigation as a national question of the mortality of special classes, as for instance the Army and the Navy. (This branch of

investigation might be extended almost indefinitely, and with great advantage to the State, particularly if certain forms of industry are nationalized hereafter).

(*d*) The question of State Pensions which for many years must be a point of anxious consideration for the State and for every citizen.

It will be seen that these are all public questions, covering a great many possible sub-divisions which I need not specify. All of us for instance can imagine many directions in which actuarial treatment might be more fully applied to the Census results. Great questions of national efficiency are involved in a complete examination of these results, including problems of trade and manufacture, taxation, national thrift, man power, whether for military or for other special purposes, and others which I will not indicate. These kinds of questions are what I had in mind when I suggested earlier in my address the possibility that the Institute and its members might prove of greater assistance to the Commonwealth in the future than they have had the opportunity of being in the past. As we know the Institute as a body and many of its individual members have given much valuable service to the State, but my present point is that if we lay ourselves out to take up these matters as part of an extended curriculum, there will naturally be submitted to us many more national questions of the kind indicated than have come before us in the past.

As regards Insurance there are directions in which actuarial science may still be applied to subjects with which we may be presumed to be more familiar already than with some of those enumerated above. In ordinary life assurance there is hardly room for much that is new, but it is convenient to refer to it here. On the purely technical side, variety and cheapness of policies, improvements in methods of valuation tending to greater expedition and accuracy, increased liberality in conditions, &c., there is probably little to be done. The general soundness of the lines on which life assurance institutions are conducted, including their finance, has, so far as we can tell at present, stood a test more severe than it was ever contemplated should be applied to them. Whatever further trials fall upon them they can hardly be so serious as those through which they have lately passed, let us hope

with complete success, although the final reckoning is yet to come.

But in one phase or incident of ordinary life assurance, the paper which shall be a classic remains to be written. The question of the proper terms on which two life assurance offices should carry out an amalgamation as distinct from a transfer, presents opportunities for wide differences of opinion and of treatment. When and why should the business of one office be kept open and the other closed? When is it advisable to keep both open and amalgamate them? In that case should existing surpluses be allocated to existing policy holders, and if not how should they be divided? On what principles should the future premiums to be charged to new entrants be determined? How should the proper method of allocating future profits between the existing policy holders of both offices and future entrants into the combined office be decided? These are questions which may quite possibly assume considerable importance in the future, and guidance upon them would be useful not only to actuaries but to others, such as directors and policy holders, who are directly interested.

In connection with one section of life assurance, industrial assurance, I think that there are possibilities of betterment into which careful enquiry should be made. The great objection to industrial assurance, in spite of the inestimable benefit which it has been to the cause of national thrift is its expense. On the lines on which this class of assurance is necessarily conducted at present there does not seem to be much hope of a reduction in cost. I would, however, commend to those who are particularly interested in it the system of "group insurance" which is making great progress in America.

This system is not to be confused with that of "collective insurance," on which the late R. P. Hardy read a paper here some years ago. His paper was designed to suggest for consideration, in connection with the question of Old Age Pensions, a system of endowment assurance or deferred annuities for the whole population. He submitted some interesting formulæ, but as a practical question the matter has lost its interest at least for the present. One thought is however forced upon me in reading the discussion, in which practically every speaker, possibly influenced by a solemn warning given in advance by the then President, felt

constrained to avoid anything which might be construed to have even a remote bearing on the political situation. The result was that the discussion consisted largely of ingenious evasions of political references or apologies for approaching them. I mention this only because I feel that if in the conditions which exist this Institute is to take its proper share in public life such an attitude towards political questions is impossible, and indeed childish. I suppose that one of the most burning political questions of the moment is that of Free Trade, but I can conceive some aspects of that matter on which actuarial investigation might throw light, while the whole question is one largely of the correct application of statistics. If my idea is correct some actuary in the near future might wish to read a paper on Free Trade versus Protection, and I sincerely trust that his enterprise would not be discouraged, and that, if it were, it would be on any grounds except the political one.

But to return to the question of "group insurance." Its central idea is that employers should provide out of their profits if possible (and in America apparently it is possible) death benefits for their employees by a premium which is calculated to provide for all the deaths taking place during one year amongst such employees. Only one policy is issued to cover the whole risk. It is a "renewable term" policy and so far as I can gather the soundness of the system depends on the fact that the numbers and age distribution of the employees in a large firm remain fairly constant. I have not seen any formulæ or indeed any reference to the mathematical basis of the system, but there is a non-technical description of it in "The Annals of the American Academy of Political and Social Science" for March 1917.

From this it appears that although the system has been introduced into America only since 1911 it has attained wide-spread favour amongst large employers. It is stated, for instance, that the Union Pacific Railroad has insured 35,000 employees each to the extent of one year's salary, the total insurance under this one policy amounting to $30,000,000 while during 1916 one insurance company issued under this plan more than $40,000,000 of insurance, and I have since heard that the Standard Oil Company has adopted the scheme for 500,000 employees. The unit of insurance recommended is one year's salary with a maximum of $3,000 and on this

basis the monthly premium is said to be approximately 1 per 1,000 or 1·2 per-cent per annum.

The system has scarcely been in force long enough to warrant any confident statement as to its soundness in the form described, but inasmuch as under the American scheme the whole of the premium is paid by the employer (and large employers of labour in America are not the least shrewd of their race) it may safely be assumed that, at any rate from his point of view, the system is satisfactory. Whether this fact implies the opposite possibilities, namely, that it is unsatisfactory to the employed or to the insurance company, I cannot say, but it is worthy of note that according to the writer of the article to which I am referring the improved conditions which result from the adoption of this insurance scheme, and the closer and more confident relations which are thereby established between employer and employed, have made employers ready to make what at first sight must appear considerable sacrifices in order to introduce it amongst their men. As I know from recent experience one of the greatest difficulties with which large employers have to contend is the continual disturbance caused by the loss and replacement of the lower grades of employee, and this difficulty is that which it is asserted that group insurance largely meets.

I am aware that these are not actuarial considerations, but they have a direct bearing on social economy. Having regard to what I have already said as to the duty of the Institute towards society and also to the fact that industrial assurance would probably be the first object of attack by an ill informed democracy anxious to nationalize insurance, it seems well worth while for actuaries, who are interested in or identified with that form of assurance, to examine this scheme and if it is sound to fit it, or possibly some improvement on it, into their own organizations. At least it will be seen that industrial assurance on these lines is enormously cheapened. As already stated the policy is a yearly renewable one, which implies a very low premium rate, as no large reserve fund need be maintained. There are practically no funds to invest and no expense is incurred in their care. There are no medical fees, and small or no commissions. The cost of collection is infinitesimal and is really borne by the employer. Naturally there is a very large saving in stationery, in accounting, and other staff expenses.

On the other hand it is obvious to us that if the plan is open to the same objections as ordinary "assessmentism" it must be unsound, and should therefore be discouraged. Like all works schemes its solvency depends on that of the employer, but there are the special objections to this particular scheme:

(1) That the first men to leave the firm which was tending towards insolvency would be the young and healthy. The renewable term premium would then be insufficient to meet claims, and would have to be raised, perhaps to an impossible figure, if it is to be adequate.

(2) That in the eventual wreck there would be no surrender values, or other form of salvage.*

As regards other forms of insurance I feel that there is still something to be done in the investigation of sickness experience with a view to the extension of sickness insurance. A large body of facts must have been accumulated in the last few years, and it seems reasonable to suppose, on *a priori* grounds, that an enquiry into these facts might permit such a cheapening of this form of insurance as would make it much more attractive than it is now to those large bodies of workers for whom a cessation of work means a cessation of earned income. To those outside National Health Insurance and particularly to the professional classes, sickness insurance should specially appeal. It would practically take the place of the unemployment insurance of the wage earner, and a more general use of it would fill one of the remaining gaps in a system of general protection against all the "bludgeonings of chance" which our present opportunities and knowledge should make complete.

That there are possibilities in the direction indicated appears to be recognized by the large accident companies and by certain life offices. The former have extended the limits within which their provision for sickness operates, while the latter provide for the suspension of premium payments during total disability from accident and certain forms of sickness, apparently without extra charge.

Accident insurance, including employer's liability, is yet another wide field for the actuary only partially opened up.

Some progress has been made in devising and enunciating correct methods of valuation, but I do not see in the suggested

* See also J. Burn, *J.I.A.*, vol. xlix, p. 229.

course of reading which accompanies the new syllabus any reference to particular books or papers on the subject, although the student is referred, with sardonic humour, to the returns to the Board of Trade which relate to the employers' liability business. This omission will no doubt be remedied. There is at least one paper on the subject (vol. xlv, p. 101, W. Penman) and possibly there are other writings of which I have no knowledge.

There must be by this time an enormous mass of material a thorough examination of which would throw light on many points, and give valuable general information to the insurance manager, the actuary, and the state. I am aware that one of the objections to the application of the actuarial science to the business of employers' liability is that in this class of insurance the operation of the laws of probability is interrupted or diverted by the independent exercise of the human will on the part of the workman, and of the judges. It seems to me, however, that in a large mass and over long periods of time this objection is either negligible or if it is not, that then its effects must be traceable and measurable, even if they cannot be distinguished from other effects having a different origin but the same result. I confess that I should like to see a comprehensive investigation of the facts relating to all classes of accident insurance accumulated by the large composite companies. The data must be extensive enough to warrant their sub-division into many heads with results which would be both interesting and informing.

Unemployment insurance I do not see mentioned in the syllabus or course of reading. It is naturally still in its infancy, but it is part of the national scheme of insurance, the correct calculation of its rates of premium is certainly an actuarial matter, and therefore it must sooner or later come within the purview of the Institute, if it can be continued at all. Of this I am doubtful. It seems to me that the human and extra-human agencies—politics, strikes, &c., on the one hand, and trade cycles on the other—are disturbing agencies too uncertain for science to measure their effect and too powerful to be neglected. The aim of society should be to lessen unemployment, or to abolish it if possible, by social and industrial reforms or by increased production, rather than to pay for it by insurance. That course is a mere shifting of the burden.

There are various other forms of insurance against minor ills or accidents—the occurrence of twins, for instance—which are usually effected at Lloyd's. All these are susceptible of treatment either actuarially or statistically, and should therefore come within the scope of the Institute's activities. A more extended investigation into the question of insurance against issue is overdue, and should prove both valuable and interesting. But all these are better suited to be the actuary's recreation than to form part of his serious pursuits.

Fire and marine insurance are hardly susceptible of actuarial treatment in the strict sense, although a consideration of their methods and results from a statistical standpoint might very well prove a profitable undertaking for an office which was large enough to provide its own facts and enterprising enough to submit them to expert examination. But that suggestion is only part of a much larger one which I should like to submit to the responsible heads of the great composite companies. It has probably occurred already to the bold and imaginative minds which have made so striking a success of British insurance, but I do not know that it has ever been carried into effect. My proposal is that the large offices should create separate statistical departments, the basis of which would be their existing actuarial departments. These departments would need enlarging and strengthening, and to them would then be referred, not only the usual actuarial work of the life branch, but the statistical examination of all other branches and of the questions arising in them. That such a department would more than pay its way I have no doubt, and if it were found possible, as I think would be the case in time, to include among its duties the constant care and scrutiny, if not the actual management, of the investments, then I am sure that it would be very profitable. The present time or the immediate future, with their growing crop of new and difficult problems, seem very suitable for the initiation of something on these lines.

Before leaving the subject of insurance I should like to refer briefly to the possibility of a new investigation into the mortality of assured lives and annuitants. It is 25 years since data were last collected, and though a general investigation, even if it were necessary, is probably out of the question, and of doubtful permanent value owing to the events of the last four years, yet I feel that some early enquiry into the

effects of the war upon the mortality of the offices is both necessary and desirable.

I have specified certain directions in which lie wider openings for the activities of the Institute and its members in connection with government work. A general consideration of existing conditions prompts the suggestion which I shall make for dealing with the situation as a whole and meeting the need for more and, except in certain departments, better actuarial assistance than has hitherto been available for the service of the State. If we read through a list of government departments we shall see that there is a very large proportion of them in which actuarial knowledge and training must be either indispensable or advantageous. Such are the National Health Insurance Commission, the General Register Office, the Registry of Friendly Societies, the National Debt Office, the War Office, Admiralty, and any other department which deals with its own pensions, or other pensions on a large scale, and the Boards of Inland Revenue and of Trade. There are certainly others in which actuarial knowledge if not indispensable is nearly so, such as the Treasury, India Office, the Colonial Office and the Local Government Board, while the proposed Ministry of Health, if and when it comes into being, must also afford an appropriate field for actuarial work and methods.

Other smaller departments—Savings Banks, Paymaster General's Department, Public Works Loan Board—would probably be the better for it, and in any case actuarial or statistical questions requiring expert advice must arise in them from time to time.

How can the needs which I have indicated best be met? I had hoped, when the appointment of Sir Alfred Watson to the post of government actuary was announced, that that appointment would be a preliminary to the constitution of a special government department of which the government actuary would be the head. This department would be one, not only of actuarial work, but of general statistical enquiry. To it would be referred the preparation and examination of the statistical data collected by all other departments in the course of their particular work, as well as the various special points which would present themselves for consideration. Such a department would necessarily be a large one and the nucleus of it must be formed by the government actuary's

existing staff (at present I imagine largely if not wholly engaged in the National Health Insurance Commission) with the actuaries now attached (whether as actuaries or not) to other government departments—the War Office, National Debt Commission, Somerset House and others.

The new section of the administration would need to be a statistical as well as an actuarial department, having regard to the mass of statistical work which would or should be submitted to it by other departments, such, for instance, as the Board of Trade. But the bulk of its work would probably come from the National Health Insurance Commission, the Ministry of Pensions, and the General Register Office, and would be actuarial. However this may be, I feel that if such a department is to be instituted it should be started and controlled by the government actuary. No other official can have had equally good training for, or so wide an experience of, both kinds of work, the statistical and the purely actuarial.

That the necessity of a government statistical department is realized in other quarters is shown by the introduction into the recent report of the Committee on the Cost of Living of the following paragraph:

"If it is not too far exceeding the bounds of our terms "of reference, we would add that we have been greatly struck "with the improvement which might be made in the value of "the large body of statistics which the different government "departments regularly collect each for itself, if some authority "existed charged with the duty of keeping them all in line, "ensuring the employment of uniform standards and inter-"related methods and unifying the whole by the application of "principles of scientific co-ordination."

This recommendation, which is all the more striking because it has only an indirect relation to the subject of the report, is supported by Professor Ashley and Professor Bowley, members of the committee whose authority in such matters is unquestioned.

There is no need to point out in detail the directions in which the new department would prove of value. Its least recommendation is that the cost of it would be infinitesimal compared with the gain to the country which must result say from competent examination of the statistics relating to the trade of our own country now handled by the Board of Trade, and of the similar statistics obtainable from foreign countries.

In one other direction to which I have already referred its advice and assistance would be of the greatest value, even if it did not actually undertake the work, now admirably performed, within the limitations imposed upon them, by the Registrar-General and his staff. It has long been a dream of actuaries and statisticians that in the perfect state there shall be a permanent census office, constantly examining and re-arranging in different lights the results of each census. It is further urged that censuses should be taken at least quinquennially, instead of decennially as at present. The Registrar-General has carried this idea even further. He has pointed out to me that with slight alterations the present machinery of national registration could be adapted to the purpose of an annual census.*

The social and economic value of a census which would be practically continuous can hardly be over-estimated, and having regard to the comparatively simple character of the main facts required, the Registrar-General's proposal seems eminently practicable. The question which department should carry out the work would no doubt form the subject of a fierce struggle between my friends Sir Bernard Mallet and Sir Alfred Watson, but whichever won the thing would be properly done and we should have got it, which after all is the main consideration.

To National Health Insurance I need hardly refer. We know it to be in good hands and it is fortunate that it is so, for the war must have created for the department a great many problems which will tax to the utmost the skill and judgment of its actuarial and financial advisers. In the approaching valuations of the approved societies, the disturbance of the rates of mortality and sickness both immediate and future, the possibility of increased benefit as a result of political and economic pressure, and the depreciation in pre-war investments, are all facts which must be faced.

I have mentioned State Pensions as a subject which will be one of anxiety to the state and to every thoughtful citizen for many years. In dealing with this matter it is necessary to speak plainly. For obvious reasons it has been taken up enthusiastically by a certain class of politicians, of all shades

* The Registrar-General has developed his suggestion in a lecture to the Royal Institute of Public Health. The lecture was published in "The Journal of State Medicine" for August 1918, and has been reprinted by Messrs. John Bale, Sons & Danielson, Ltd., 83-91, Great Titchfield Street, London, W.1.

of opinion, who vie with each other in extravagant promises to the electorate, and in applying pressure to a facile minister. To the lazy and self-seeking nothing is so easy as to be generous with the money of the state, and nothing requires more courage than to resist a demand when resistance lends itself to interested misrepresentation. Already we have seen one pliant minister succeeded by another still more obliging, while official publications contain many protests against laxity of administration by both, and warnings as to its consequences.* The Government, since the dissolution and therefore without consulting parliament, has increased the scale of pensions by 20 per-cent until June next. "If they "do these things in a green tree what shall be done in the "dry?" With the example of the United States before him the ordinary taxpayer may well ask the question, while we whose special knowledge and experience tell us how insidiously the burden of pensions grows and how heavy it can become, cannot but regard the whole matter with profound anxiety. In my view it is absolutely essential, both on actuarial and political grounds, that the administration of War Pensions should be removed, as far as possible, and if possible entirely, from the influence of the government and of parliament. It is greatly to be hoped that in the new parliament there may be a body of members sufficiently independent to adopt this view and strong enough to enforce it. But in this respect a study of election addresses is decidedly discouraging.

As a result of the war and of the gathering interest in social problems the question of Superannuation Funds is likely to assume considerable importance in the near future. Existing Funds will need careful examination and the most expert management if they are to escape disaster and fulfil their useful functions. No one can foresee the exact effect of the war on the rates of mortality and retirement in these funds, but we know that some of the direct results may be (*a*) heavy payments of death benefits in respect of young members; (*b*) earlier retirement on pension as a result of disability or invalidity; (*c*) an increase in the future mortality of pensioners. There are indirect results to which my

* These protests and warnings are too long to quote, but the members may be referred to (*a*) the report by the Government Actuary, paragraph 6 appended to Cd. 8485, 26 February 1917; (*b*) The 2nd Report of the Committee on National Expenditure No. 30 of 1918; (*c*) Report by the Government Actuary, paragraph 5, Cd. 9054, 18 April 1918; (*d*) Report of Committee of Public Accounts, No. 1000 of 1918, pp. xii and xiii and 227, &c.; (*e*) Ninth Report of Committee on National Expenditure No. 121 of 1918, paragraphs 27 and 29 (15).

attention has been called in connection with one very important fund and these I should like to bring to your notice. In the first place there is a not unnatural agitation for an increase in the pension of those annuitants who have retired before or in the early days of the war and now find that a pension based on pre-war conditions of pay and prices is wholly inadequate to present needs. The constitution of these superannuation funds is generally very democratic. The rules as to benefits are easily altered, but in my experience it is the rarest thing to find that the funds will stand any increase in benefits, while in many cases a diminution is more or less urgently needed. Again, in those funds where the pension is a function of the final salary or of the average salary of the last few years, to base the pension on the increased salaries or wages which have been given during and in consequence of the war will upset the whole foundation of the funds. Then there is the universal depreciation in securities to be faced, although there is compensation for this in the higher rate of interest which seems likely to prevail for some years.

These points cannot be elaborated here, but they and many others deserve the consideration of all those who are interested in these funds. The matter will become of special interest if the railways are nationalized because in that case it is to be presumed that the government will assume the responsibility for the funds connected with them. Those of us who remember the Report of the Departmental Committee on Railway Superannuation Funds which appeared in 1910 will remember the condition of many of them, and how several of the railway companies dealt with the position by guaranteeing the solvency of their funds. Unless the government take over this liability, which will then be a charge on the taxpayer, the amount of it must be deducted from the purchase price. In that case the value of the junior securities of the companies interested may be somewhat seriously affected.

I have now detailed the main directions in which, as I think, the services of the Institute and of its members can be further utilized. But if they are to be employed to the best advantage then it is necessary that additional training to that which the new curriculum affords should be provided. The members of the Institute will agree with me that if it is in any way possible this training should be given within our walls. I believe that it is possible and in this belief I venture to outline the following scheme.

There are five subjects which seem to me suitable to form the substance of what for convenience I will call post-graduate courses. They are:

(*a*) General statistics, including graduation in its more scientific developments.

(*b*) National Health Insurance and Unemployment Insurance if unemployment is found to be susceptible of insurance.

(*c*) Friendly Societies, Pension Funds, and Widows' and Orphans' Funds.

(*d*) Law.

(*e*) Economics and finance.

I see no reason why all these subjects should not be taught within this building and all, except possibly the last, by members of our own body. If this is feasible then the objects at which I am aiming will be completely attained.

If I may explain my idea rather more fully it is this:

(1) To enlarge the boundaries of the Institute's work and training so as to cover not only those fields (principally insurance and various forms of thrift) which it has hitherto cultivated, but also some others not now within its special domain.

(2) To secure that our members are properly trained for this work or for one or two branches of it, and do not, as too often in the past, acquire merely a smattering of all branches.

(3) To provide means by which ambitious students may fit themselves for other employment than that to which in practice they have so far confined themselves.

(4) To insure that those whom the Institute has trained, should remain members of it, whatever career they may ultimately select.

I have already referred to the difficulty of finding a definition of an actuary, and the existence of that difficulty implies that the limits of our work are not, even now, very strictly marked off. There can therefore be no objection to enlarging them, even at the expense of sacrificing all existing definitions, including my own. This will at least create opportunities for further ingenuity, and I am not sure that the definition which I have criticized, "a scientific financier" would not hereafter apply with fair accuracy to the actuary who qualified under the syllabus and subsequently took honours in statistics, economics and finance. In any case I cannot conceive any man who should have a better claim to the title than such an actuary.

As I said before, there is no reason why the first four subjects should not be taught here and by our own members. Some persons unaware of the facts might object to the Institute attempting to teach law, but these I would remind that we have among us many barristers, who although they do not practice in the courts, are constantly required to examine various aspects of conveyancing and commercial law, who thus acquire a more general knowledge than many practising barristers, and are certainly capable of imparting it to others. Indeed, if our members are to be taught law we may have to do it ourselves, for I am told on enquiry that the latest regulations might possibly be construed as barring admission to an Inn of Court to actuaries, as to members of certain other professions such as chartered accountants.

As regards economics and finance, I have expressed some doubt whether we could provide lecturers from our own body. Personally I do not think that we should be able to do so entirely, although there are certain aspects of finance, including the choice and care of investments, in which the experience of some of our members is as wide as that of anyone. It includes one class of investments, reversions and life interests, which only actuaries are competent to handle, while we may not unfairly claim that our general knowledge of all the incidents attaching to mortgages is at least equal to that of any other professional body, except lawyers.

But as regards Stock Exchange securities I cannot help feeling that we are not sufficiently in touch, not only with the influences, personal and otherwise, which produce temporary variations in values, but also with those factors and tendencies which make for changes more general and permanent in character, and, on that account, are specially important. But although we cannot supply from amongst us lecturers on all the subjects included under the wide heading of this section it is quite possible that we may be able to find teachers for some of them. For the rest we might rely on the admirable course of the London School of Economics; and in taking this course students have the opportunity of becoming at the same time graduates of London University. It might be feasible, if we can guarantee enough students, for the professors of the School of Economics to lecture here, a course which most of us would prefer if it could be arranged, but we must avoid if possible the overlapping which would be inevitable if the Institute became a separate teaching centre for the same subjects as are now so well taught by the

School of Economics. I confess, however, that I do not at present see how to meet this difficulty.

I attach very great importance to the question of financial education. I am prepared to maintain that the most important function of the administration of a life office is the management of its funds, whether that management be actually vested in the board, a finance committee or the manager. The highest prizes which actuaries, speaking generally, have been able to reach have hitherto been the chief positions in life assurance offices, or in the life branches of the composite companies. Apart from character and personality the attainment of these positions is due in my view not so much to purely actuarial knowledge as to the general training provided by the Institute, and particularly the financial training, inadequate as this has been. How many actuaries who are also chief officers of their companies concern themselves with more than general supervision of the actuarial work? None, I imagine, and although my own opinion is that the chief officer of a life office should be an actuary, this claim must be justified in the view of those responsible for such appointments, and if actuaries are to retain their present almost universal control of life offices, it must be on the strength of wider qualifications than the purely actuarial. It follows, if my views are right, that the complete omission of financial questions from the syllabus imposes on the Council, as the trustees of the general interests of the members, the duty of providing training in finance outside the syllabus. In my opinion scientific finance may quite properly be included in the domain of the actuary, particularly if we assume, as I do, that economics and general statistics will in future also form part of his training.

In that case not only will the actuary retain unchallenged and indeed unchallengeable his position in the insurance world, but, having regard to the character of his training in other respects, I can see no reason why, if he likes to leave that world and possesses the personal qualities indispensable to success in any of the higher walks of life, he should not be capable of treading them with the best. But as I heard Mr. Arthur Balfour say recently: "No education in the "world will make up for the original deficiency, when there is "a deficiency, in the raw material to be educated. Nothing "will compensate for the absence of mother wit. There is no "substitute for that energy, originality, tact, insight, courage

"and enterprise which, in commerce as in every other walk of "life, are sure and certain instruments of success."

In this connection there is one further point to which I must refer, namely, the difficulty, however good our theoretical teaching may be, of affording opportunities for practical training. But this difficulty should not be insurmountable if directors and managers regard the movement with sympathy and understanding. They can, for instance, allow members of their staffs who are specializing in finance to attend finance committees or meetings of the board, inviting them to submit opinions and suggestions and to join in discussions. No harm could come from such association, while the value of the investment department as a source of profit must be increased. Individual managers might allow students in their offices access to the investment ledgers and encourage them to make enquiries and suggestions in regard to particular securities. They might also submit to them the offers which daily come before them, asking for a reasoned expression of opinion as to their merits. I believe that the managers would derive at least as much benefit as the students from this course. To assist and encourage the personal contact which is so necessary an element in successful finance, intelligent and tactful students should be allowed to call from time to time on the company's financial connections and urged to make such connections for themselves.

If these "postgraduate" courses are instituted, as I hope that they will be, then the Council or the Board of Examiners must prescribe appropriate courses of reading, and in my view it is essential to its success that the efficiency of the system should be tested by examinations. No one believes that examinations are an infallible test of knowledge, but they put to some proof both it and the training for it. It should be easy to devise a diploma which in the candidate's hands would be witness to the fact that he had gained honours in one or more of the additional subjects, while distinguishing marks or letters could be attached to his name in our annual list of members. If in financial matters we had to rely entirely on the training provided by the London School of Economics then their examinations, and the degree of the London University, would meet the needs which I have indicated.

The course of these remarks leads me to suggest for your consideration whether it is possible and if so whether it is

desirable that the Institute should itself become attached to London University. There are naturally difficulties in the way—the question of matriculation for example—but I see no reason why the influence and importance of the Institute should be in any way diminished by association with the University—rather the reverse. I do not know what are the exact relations between the London School of Economics and the University, but apparently it would not be impossible to reconcile incorporation in the University with the maintenance of the Institute's independent existence and training, while it might be arranged that the University degree, whatever form it took, should be an addition to and not a substitution for our own F.I.A. However, this is a large subject and I have not sufficient knowledge of the details involved to be competent to discuss it. Nevertheless I think that it should be considered, particularly as it seems to me that it is in the direction of co-ordination and centralization of teaching that the times are moving and that, if this is so, then we must go with them or risk losing our position and influence.

Since these words were written the movement towards the institution of degrees in Commerce by the London University inaugurated in July has advanced to the stage that the insurance community has been directly approached by representatives of the University with a request for such co-operation and financial assistance as has already been promised by the banking and other interests. The subject was recently discussed at a meeting at which the University was strongly represented and was referred for consideration to a committee to be nominated by the chairman of the British Insurance Association, Mr. Roger Owen. It is therefore premature to discuss it, but in the draft scheme of examinations for these degrees insurance is only one of five subjects to be covered by one paper. As the remaining subjects for this single paper are (*a*) Commercial Law, (*b*) Sale of Goods, (*c*) Carriage of Goods by Land and Sea, and (*d*) Elements of the Law relating to Negotiable Instruments, Partnership and Companies, and as about 20 papers have to be taken for the full degree it is obviously impossible that either the teaching of insurance or the test by examination can be sufficient for any practical purpose. On the other hand the full course for the degree is so extensive that it does not seem that students of the insurance institutes would

be able to take it concurrently with their present training, while for actuarial students it would be impossible. Nevertheless if the scheme is ultimately put into practical shape I think that we may assure its promoters that any help which the Institute can give by way of advice or assistance in its special subjects will be forthcoming.*

This concludes the main portion of my address. The suggestions which I have made might have been thought revolutionary a few years ago, but to us who have been tried in the fires of these last four years a revolution, especially one that is purely domestic, is a small matter. For this reason I commend my proposals to your consideration with the assurance in my own mind that they will be examined and discussed on their merits, and with the sympathy which is due, not to your President or to me, but to any well meant effort to improve the world in which we move, and that particular part of it which is the scene of our own endeavours. Frankly, though the main object which I had in mind in writing this address was to suggest some ways in which the status of the Institute and the profession might be raised and their scope enlarged, yet I hope that in doing this we may increase the value and therefore the emoluments of the individual actuary. I have no patience with the cant which affects to believe that all our efforts should be purely unselfish. We have seen and heard a great deal of it recently in connection with national and international affairs, but I do not notice that certain classes of the community in pursuing their ideals have overlooked their own interests, even though their action seriously endangered the objects which they profess to have in view. Private gain is not incompatible with public good. On the contrary the better the man the more should he be worth not only to himself but also to the community. But within the limits of his opportunities he must work as hard for the community as for himself. Only thus shall he earn, and even possibly receive, the full measure of his reward.

I must ask you to bear with me for yet a few minutes while I refer to a question which, though not at present a burning one, is certainly smouldering, and therefore requires examination and careful handling. I mean the admission of women to the Institute. I am not one of those who think

* On this subject see a letter from Mr. S. G. Warner, which appeared in the *Insurance Record* of 20 December 1918.

that because women have done nobly in the war, as they have, that therefore they should be invited to undertake all sorts of work and responsibilities for which they are fitted neither by temperament nor by training. Women's brains and physical capacity are no greater and no less now than before the war, but the war has given them the opportunity to prove their precise worth in both these respects. In my opinion they have proved it to be sufficiently high to entitle them to admission, on equal terms with men, to many spheres of activity hitherto closed to them. In the past the refusal so to admit them was due to want of knowledge which has now been supplied, or to prejudice on the part of men, which amounted to a denial of social justice. I do not see therefore why the Institute should affect to be deaf, or should bar its doors when women begin to knock at them. It is unlikely that any but women of high intelligence will seek admission here, and for intellect, whether it resides in man or woman, there is always room in every profession. For these reasons my personal opinion is that women should be admitted to the Institute on equal terms with men, and that if the members decide that this should be done, it should be done completely at the appointed time. I mention this because it has been suggested that for the present women should be admitted only to sit for the examinations. To me this looks like fencing with a question which must be faced. We may be certain that the demand for full admission will not be suppressed, even if at their first examination all the women candidates were to fail, and it is better to make a concession as a result of considered judgment than to yield to importunity.

As I have said, these are only my personal opinions. Officially I may say that while there is no reason why the question should not be discussed at this time, yet there are good and obvious reasons why it should not be decided before the return to our hall of those of our members who have been forth to fight. If, therefore, the matter is pressed I should use such influence as your President may possess and deserve, to prevent action the result of which might be regarded, in existing circumstances, as an injustice, either by absent members on the one hand, or by aspiring women on the other.

Newton's Interpolation Formulas. Further Notes by DUNCAN C. FRASER, M.A., F.I.A.

THE following references to the subject will be found in the "Commercium Epistolicum":

Letter from Leibnitz to Oldenburg, dated 3 *February* 1672/3. —The "Commercium Epistolicum" includes ten letters from Leibnitz to Oldenburg, and references to five others, besides four letters from Leibnitz to other mathematicians.

Only one of these, the earliest of the series, discusses questions relating to Finite Differences; and as it is entirely devoted to this subject and furnishes an interesting indication of the state of knowledge at the time, I have given a version of it in full. It will be observed that while the heading clearly states that it is addressed to Oldenburg, he is referred to in the course of the letter in the third person and by name. It seems to have the form of a memorandum rather than of a letter, and we may suppose it to have been written for private circulation, or possibly with a view to publication in the *Transactions* of the Royal Society, of which Oldenburg was Secretary.

The paragraph in which Leibnitz explains a general rule relating to the differences of any kind of powers is merely a verbal description of the formula

$$a^n - b^n = a^{n-1}(a-b) + b(a^{n-1} - b^{n-1}).$$

In his final paragraph he gives some examples of series with fractional terms, the general term of the rth series being

$$\frac{\underline{|r}}{(n+2)(n+3)\ .\ .\ .\ (n+r+1)}.$$

The rest of the letter is taken up with a discussion of the fundamental formula of Finite Differences, $u_n = (1+\Delta)^n u_0$, for the case when n is an integer; and with remarks on the properties of the coefficients which are employed.

It appears that Leibnitz and writers before him found no difficulty in writing down the formula for any given integral value of n, the necessary coefficients being obtained by reference to a table. Leibnitz employed a table drawn from Pascal's Treatise on the Arithmetical Triangle (printed in 1654, but not published until 1665); an account of which is given in the "History of the Theory of Probability" by Isaac

Todhunter, who mentions the application of the Arithmetical Triangle by Pascal in solving questions of Combinations and Probabilities, and in obtaining the powers [*i.e.*, integral powers] of binomial quantities. The particular point of interest to us in the present connection is that mathematicians before Newton were in the habit of using one and the same table for obtaining the integral powers of binomial quantities, and for expressing any one of a series of values in terms of the initial value and its leading differences.

In vol. xiv of the *Journal* (pp. 1 and 73) references will be found to the early use of methods of Finite Differences by Briggs and by Mouton, both of whom are mentioned in Leibnitz' letter.

Newton's letter of 13 *June* 1676.—This letter needs little comment. In it the first enunciation appears of what we now call the Binomial Theorem, a name not used by Newton; and he gives detailed examples of its application. A series is also given for finding a number from its logarithm, *i.e.*, the exponential series. There are some remarks of a vague and general character on methods of obtaining approximate series which probably refer to methods of differences. In my version I have omitted a short section on the roots of equations.

Newton's letter of 24 *October* 1676.—This letter constitutes the principal document in the controversy between Newton and Leibnitz as to priority in the discovery of the Differential Calculus. The letter is a long one; and in my version I have omitted more than half of it, retaining only those sections (fortunately non-controversial) which contain matter of interest in relation to the subject of interpolation by methods of finite differences. It opens with a clear and detailed account of the process of discovery of the Binomial Theorem; and it is interesting to note that the discovery arose out of a problem in interpolation, the question which Newton set himself to solve being to ascertain the form of the series which would represent the function $\int (1-x^2)^m \, . \, dx$ for a fractional value of m, the forms of the series for a number of integral values of m being known. This was an example of the "interpolation of series", *i.e.*, of the insertion of a new series among a number of known series. The solution of this question suggested the solution of the simpler problem of finding the general form of the series for the

expansion of $(1-x^2)^m$, for any value of m integral or fractional, which led at once to the Binomial Theorem; and the Binomial Theorem in turn suggested many of the familiar processes which are included in the general description of "Algebra up to the Binomial Theorem."

In a letter dated 21 June 1677 to Oldenburg, Leibnitz says, "His description of the way in which he was led to some of "his very elegant theorems is singularly happy; and what he "says on the interpolations of Wallis is especially pleasing, "because by this argument a proof of these interpolations is "obtained which (so far as I know) had previously been given "by induction only."

The date of discovery of the Binomial Theorem is fixed by a note left by Newton in which he says: "In the beginning "of the year 1665 I found the method of approximating series "and the rule for reducing any dignity " [*i.e.*, power] " of a "binomial into such a series."

Newton's proof depends solely on the properties of the coefficients, and is not affected in any way by the nature of the quantities, algebraical or otherwise, with which they are associated. Remembering that the mathematicians of the time were in the habit of familiarly using the same table of coefficients for obtaining the expansions of the integral powers of binomials, and for expressing any one of a series of known values in terms of the initial value and its leading differences, it will be appreciated that Newton's discovery of a general formula for the expansion of the fractional powers of binomials gave him command at the same time of the fundamental formula of interpolation $u_n=(1+\Delta)^n . u_0$, which he embodied more than twenty years later in Lemma V of Book III of the Principia.

That he at once proceeded to apply his theorem in this direction is suggested by the fact that in the summer of the same year he engaged in extensive calculations of logarithms. In two separate sections of the letter he gives elaborate details of his methods, and in a note dated 4 July 1699, quoted in Brewster's Life of Newton, he says: "In summer 1665, being "forced from Cambridge by the plague, I computed the area "of the hyperbola at Boothby, in Lincolnshire, to two and fifty "figures." The MS. is still preserved in the University Library at Cambridge, and is mentioned in the "Catalogue "of the Portsmouth collection of books and papers written

"by or belonging to Sir Isaac Newton" (Cambridge, 1888), under the title "Calculation of the Area of the Hyberbola", being item No. 4 of sub-section "Early papers by Newton. (Holograph)." An examination of this MS. and of other MSS. in the same collection, for example the "Regula Differentiarum", which is No. 5 of sub-section "Miscellaneous Mathematical subjects" might bring to light valuable information.

Newton remarks in his letter that at a later date he used other methods which gave logarithms more exactly, and the editors of the "Commercium Epistolicum" refer to the "Geometria Analytica." This work was first published in 1736 in a translation by Colson and it contains formulas for the interpolation of logarithms derived from the properties of the logarithmic series.

The "Logarithmotechnia, sive methodus construendi logarithma nova, accurata et facilis" (4to. London), of Nicholas Mercator (not Mercator of the Maps), which is mentioned by Newton was published in September 1668. In this work Mercator took the equation to the hyperbola in the form $y=\frac{1}{1+x}$; and by simple division, explaining each step of the process in great detail, he obtained the series $1-x+x^2-x^3+$, &c., the integration of which term by term gave the logarithmic series. This was the first time that the operation of division with algebraical symbols had appeared in print and it excited extraordinary interest among mathematicians, though as the editors of the "Commercium Epistolicum" explain it was already known that the expression $\frac{1}{1-x}$, x being less than unity, represented the sum of the series $1+x+x^2+x^3+$, &c., taken to infinity. On his attention being drawn to this publication, Barrow, then Professor of Mathematics at Cambridge, made it known that a young friend of his, by name Newton, had previously arrived at general propositions of which Mercator's example was only a particular case.

Newton mentions that he wrote some papers on series in 1671. Wallis states that these papers, or some of them, were destroyed by fire. Some portions of them appear to have been included in the "Geometria Analytica."

In a following section of the letter will be found the references, which I have already quoted, to the "Methodus Differentialis."

In the second of the two sections on the subject of logarithms, the process of interpolation by intervals of one-tenth is repeatedly mentioned, and there can be little doubt that this is the process described in detail in the letter of 8 May 1675 to J. Smith. We may reasonably suppose that the same process is referred to in the final paragraph of my version where Newton says that he had almost decided to describe his method of inserting intermediate terms in the construction of trigonometrical and other tables.

It is a remarkable conclusion that as early as the year 1665, when he was still under the age of 23, Newton appears to have had at his command practically all the methods and facilities of computation which are now in use, with the exception of calculating machines; and when we consider that we owe the originating germ of such machines to Newton's contemporary Pascal, we may realize what a direct and vital connection there is between the ideas of the mathematicians of the 17th century and the practical work of the present-day Actuary.

Extract from Letter of February 1672/3.

Leibnitz to Oldenburg.

Recently when I happened to meet the eminent mathematician, Pell, at the house of the famous Boyle, we began to talk about numbers, and I was reminded by our conversation that I had a method of my own of constructing the terms of a series of any kind, either increasing or decreasing continuously, by a class of differences which I call generating differences. If the differences of a given series are found, and the differences of the differences, and the differences arising from the differences of the differences, &c.; and if a series be constructed consisting of the first term, and the first difference, and the first difference of the differences, and the first difference arising from the differences of the differences, &c., that will be the series of generating differences; so that if the continuously increasing or decreasing series be a, b, c, d, then putting $\backsim$ as the sign of the difference the generating differences will be:

$$1.\ a \qquad 2.\ a \backsim b \qquad 3.\ \overline{a \backsim b} \backsim \overline{b \backsim c} \qquad 4.\ \overline{\overline{a \backsim b} \backsim \overline{b \backsim c}} \backsim \overline{\overline{b \backsim c} \backsim \overline{c \backsim d}}$$

$$4. \qquad \overline{\overline{a \backsim b} \backsim \overline{b \backsim c}} \backsim \overline{\overline{b \backsim c} \backsim \overline{c \backsim d}}$$

$$3. \qquad \overline{a \backsim b} \backsim \overline{b \backsim c} \qquad\qquad \overline{b \backsim c} \backsim \overline{c \backsim d}$$

$$2. \qquad a \backsim b \qquad\qquad b \backsim c \qquad\qquad c \backsim d$$

$$1. \qquad a \qquad\qquad b \qquad\qquad c \qquad\qquad d$$

Or, in numbers, if the series be the series of cubes increasing in succession from unity, the generating differences will be 0, 1, 6, 6. I call them by this name, because the terms of the series are produced from them when multiplied in a particular way. Their use appears to be greatest when the generating differences are finite in number, but the terms of the series infinite ; as in the example proposed of cube numbers :

				0		0		0				
			6		6		6		6			
		6		12		18		24		30		
	1		7		19		37		61		91	
0		1		8		27		64		125		216

When the eminent Pell heard this, he replied that it had already been described by Mouton, Canon of Leyden, from the observation of the most noble Françis Reynald of Leyden, a man long celebrated in the literary world, in a book of Mouton on the apparent diameters of the Sun and Moon. From a letter of Reynald's addressed to Monconisius, and from a diary of the journeys of Monconisius, I had become acquainted with the name of Mouton, and the two works he had in view ; namely, the apparent diameters of the luminaries, and a scheme for transmitting the measures of things to posterity ; but I did not know that the book had actually appeared. Wherefore I hurried off at once to Oldenburg, the Secretary of the Royal Society, and borrowed the book and found that Pell had spoken truly. But nevertheless I thought that I ought to take pains that no suspicion should remain in their minds of my having wished to appropriate the credit of another man's ideas by suppressing the name of the discoverer, and I hope it will be apparent that I am not in such want of ideas as to be compelled to pilfer those of others. Moreover, I shall vindicate my honour by two arguments ; first by showing my rough notes, in which not only my discovery but also the manner and the occasion of the discovery appear ; and then by adding some things of the greatest importance not remarked by Reynald and Mouton, which it is not very likely should have been contrived by me in a night, and which could not reasonably be expected to be produced by a mere transcriber.

From my papers it appears that the occasion of the discovery was as follows : I was seeking a method of finding the differences of every kind of powers ; just as it is known that the differences of the square numbers are the odd numbers ; and I had found a general rule of this kind.

The preceding power of a given order being known, to find the power following (or the reverse) at a given distance, that is the powers of given terms ; or to find the differences of the powers of a given order, whatever their distances apart. In the powers of the

next lower order, let the power of the greater term be multiplied by the difference of the terms; and let the difference of the powers (still in the next lower order) be multiplied by the smaller term. The sum of the products will be the required difference of the powers of the given terms. I had adapted the same rule in such a way that to ascertain the powers of the terms for a higher order it was sufficient to know the powers of the given terms for any lower order. And I showed that what is observed to be the case for squares, namely, that their differences are the odd numbers, is not outside the basis of the rule proposed.

My mind being fixed on these ideas, as in the case of square numbers the differences are the odd numbers, so also I enquired what might be the differences of the cubes; and since these appeared to be irregular I sought the differences of the differences, until I found the third differences to be all sixes. This observation produced another. For I saw that the terms and the successive differences were generated from the preceding differences in the same way as all the successive terms arise from the primary differences, which I call on that account the generating differences, namely, in this case 0, 1, 6, 6. Having come to this conclusion it remained to find by what kind of addition or multiplication, or combination of these, the successive terms could be produced from the generating differences. And thus by solution and experiment, I perceived the first term, 0, to be composed of the first generating difference, 0, taken once or by itself; the second term, 1, to be composed of the first generating difference, 0, taken once; and the second, 1, taken once; the third term, 8, of the first generating difference, 0, taken once; the second, 1, taken twice; and the third, 6, taken once; for

$$0 \times 1 + 1 \times 2 + 6 \times 1 = 8;$$

the fourth term, 27, of the first generating difference, 0, taken once; the second, 1, taken three times; the third, 6, taken three times; and the fourth, 6, taken once; for

$$0 \times 1 + 1 \times 3 + 6 \times 3 + 6 \times 1 = 27;$$

and further calculation proved to me that this was general. This was the occasion of my observation, far otherwise from Mouton's way of approaching it; who happened upon this convenient method of calculation along with Reynald, when he was at work on the construction of his tables. Nor should either he or Reynald have any less praise because Briggs also had in some degree turned his attention to certain methods of this kind in his logarithmic tables, as Pell observes. For me, this much remains; that I may add some things not remarked by them so as to avoid the reputation of being a transcriber merely; for in the commonwealth of knowledge it does not matter who made an observation; the thing that matters is what was observed. First then I direct attention to a question

which is not noticed in Mouton's works, and yet is the head of the whole matter, namely, what are those numbers of which he gives a table to be continued to infinity, by the multiplication of which into the generating differences and by combining the products, the terms of the series may be produced. For you may see, from the very way in which the table is set out on p. 385 of his book, that it has not been sufficiently examined by him; for otherwise it is likely that the table would have been set forth in such a way that the connection and harmony of its numbers would be apparent; unless one is to say that he has been at pains to conceal it; for a part of the table is as follows:

1	1					
2	1	1				
3	1	2	1			
4	1	3	3	1		
5	1	4	6	4	1	
6	1	5	10	10	5	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252

It appears from this table that the relationship of correspondence of the generating numbers is only with the number of the term generated; so that when the term is the fourth it is produced from the first difference taken once, the second difference taken three times, the third taken three times, and the fourth once; and therefore in the same transverse line (4) are placed the numbers 1, 3, 3, 1. But the author has either not observed, or if he has observed it he has concealed that he knew the correspondence of the numbers if they are arranged in columns proceeding from the top downwards in the following manner:

1	1					
2	1	1				
3	1	2	1			
4	1	3	3	1		
5	1	4	6	4	1	
6	1	5	10	10	5	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252

For in this way their real and genuine nature and origin are apparent; that they are in fact the numbers, which I am accustomed to call combinatorial numbers, of which I have written at length in my dissertation on the art of combination, and which others call the numerical orders; unities in the first column; natural numbers in the second column; triangular numbers in the third column; pyramidal numbers in the fourth; triangulo-triangular in the fifth, &c., of which a whole treatise of Pascal's deals under the title of the Arithmetical Triangle; in which nevertheless I have wondered that such a conspicuous and natural property of these numbers has not been observed.

[*Note by Editors of the "Commercium Epistolicum."*—On the contrary, it has been observed. See Pascal's Arithmetical Triangle, published in Paris, in the year 1665, p. 2, where the last definition but one is this:

"The number in each cell is equal to that of the cell which precedes it in "the perpendicular column, added to that of the cell which precedes it in its "parallel column. So the cell F, that is the number in the cell F, is equal to "the cell C plus the cell E; and similarly for the other cells."]

But there is indeed an element of fortune in discovery, which does not always offer the best things to the greatest abilities, but often gives some of them to moderate abilities.

Hence the true nature of these numbers and the construction of the table is perceived, whether concealed or not by Mouton or by Reynald; for any given term of a given column is composed of the preceding term in the same column and of that in the previous column; and it also appears, that it is not a work involving any

troublesome calculation to continue the table set forth by Mouton as he demands, since these series of numbers are now everywhere described and used in calculation.

Moreover, from the observation of Mouton for interpolating proportional means between two extremes, I drew the conclusion that it could be used for continuing the extreme numbers themselves to infinity. He found a use for the rule only when the ultimate differences vanish, or almost vanish; I, however, detected innumerable cases, included in a rule which had been overlooked, where although the differences do not vanish I can from given finite numbers multiplied in a certain way, produce the numbers to infinity of very many series.

From the same foundations, I can work out many problems in progressions either in integers or fractions. For I can add and subtract progressions, and even multiply and divide them, and that very conveniently:

$$\begin{array}{cccc}
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\[2ex]
\frac{1}{6} & \frac{1}{10} & \frac{1}{15} & \frac{1}{21} \\[2ex]
\frac{1}{10} & \frac{1}{20} & \frac{1}{35} & \frac{1}{56} \\[2ex]
\frac{1}{15} & \frac{1}{35} & \frac{1}{70} & \frac{1}{126} \\[2ex]
\&c. & \&c. & \&c. & \&c.
\end{array}$$

Many other points about these numbers have been noticed by me, of which the above is one of the most important; I have a method of finding the sum of a series of fractions decreasing to infinity, of which the numerator is unity, and the denominators are the triangular or pyramidal numbers, or triangulo-triangular numbers, &c.

Letter of 13 June 1676.

Newton to Oldenburg.

(To be communicated to Leibnitz.)

Although the modesty of Leibnitz, in the extracts which you have lately sent me from his letter, attributes great credit to our countrymen for their investigations into Infinite series of which there begins to be talk; nevertheless I feel no doubt that he has not only discovered, as he claims, a method of reducing all sorts of quantities into series of this kind, but also a variety of convenient processes very like ours if not even better.

But since he wishes to know what discoveries have been made in England in the subject, and it happens that several years ago

I engaged in investigations of this kind, I have sent you some points that have occurred to me so that I might meet his wishes at least in some degree.

Fractions are reduced into infinite series by division, and radical quantities by the extraction of roots; by performing these operations on symbols in the same way as they are usually performed on decimal numbers. These are the fundamental principles of such reductions.

But the extractions of roots are greatly shortened by this theorem

$$\overline{P+P.Q.}\Big|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}A.Q. + \frac{m-n}{2n}B.Q. + \frac{m-2n}{3n}C.Q$$
$$+ \frac{m-3n}{4n}D.Q + \&c.$$

where $P+P.Q.$ signifies the quantity of which a root, or a power, or a root of a power is to be found; P is the first term; Q the remaining terms divided by the first. Also $\frac{m}{n}$ is the numerical index of the power of $P+P.Q.$ whether the power be integral or fractional, positive or negative. For as mathematicians are accustomed to write a^2, a^3, &c., for $a.a$, $a.a.a$, &c., so for $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{c.a^5}$, &c., I write $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$, &c., and for $\frac{1}{a}$, $\frac{1}{a.a}$, $\frac{1}{a.a.a}$, I write a^{-1}, a^{-2}, a^{-3}. And in the same way for $\frac{a.a}{\sqrt{c.}\,\overline{a^3+b.b.x}}$ I write $a.a\times\overline{a^3+b.b.x}\Big|^{-\frac{1}{3}}$ and for $\frac{a.a.b}{\sqrt{c:(a^3+b.b.x)\times(a^3+b.b.x)}}$ I write $a.a.b\times\overline{a^3+b.b.x}\Big|^{-\frac{2}{3}}$. In the last case if $\overline{a^3+b.b.x}\Big|^{-\frac{2}{3}}$ be taken to stand for $\overline{P+P.Q.}\Big|^{\frac{m}{n}}$ in the formula, P will be $=a^3$; Q will be $=\frac{b.b.x}{a^3}$; $m=-2$, $n=3$. Finally for the terms found in the quotient in the course of the work, I employ the symbols A, B, C, D, &c.; namely, A for the first term, $P^{\frac{m}{n}}$; B for the second, $\frac{m}{n}\cdot A.Q$; and so on in succession. In other respects the use of the formula will be plain from examples.

[The examples, each of which is expanded to several terms, are

(1) $(c^2+x^2)^{\frac{1}{2}}$

(2) $(c^5+c^4x-x^5)^{\frac{1}{5}}$. Here P may be taken as c^5 or as $-x^5$. "The former is to be preferred if x is very small; the latter if x is very great."

(3) $N\times\overline{y^3-a^2y}\Big|^{-\frac{1}{2}}$

(4) $(d+e)^{\frac{4}{3}}$

(5) "In the same way simple powers also are produced."

$$\text{Ex}: (d+e)^5$$

(6) "Moreover, Division, whether simple or repeated, is accomplished by the same rule."

$$\text{Ex}: \frac{1}{d+e} = (d+e)^{-1}$$

(7) $(d+e)^{-3}$

(8) $\mathrm{N} \times (d+e)^{-\frac{1}{3}}$

(9) $\mathrm{N} \times (d+e)^{-\frac{3}{5}}$]

By the same rule, expansions of powers, divisions by powers or by radical quantities, and extractions of the higher roots of numbers, are also conveniently performed.

* * * * *

It would take too long to describe how, from equations so reduced to infinite series, the areas and the lengths of curves, the volumes and the surfaces of solids, or of any segments of any figures, and their centres of gravity, may be determined; and also how all mechanical curves can be reduced to equations of infinite series of this kind, and problems concerning them resolved just as if they were geometrical curves. It may suffice to review some specimens of such problems; and in these I shall sometimes for the sake of brevity use the letters A, B, C, D, &c., to indicate the successive terms of a series taken from the beginning.

(Nine problems are discussed. The first three relate to trigonometrical functions, and the next three to the ellipse; the eighth to the quadratrix, and the ninth to the spheroid. The seventh is as follows):

(7) if C D be an hyperbola, whose asymptotes EB, EF make the right angle BEF; and if on EB are erected perpendiculars AC, BD, meeting the hyperbola in C and D; and if EA be called a; AC, b; and the area CADB, z;

$$\text{AB will be} = \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{120a^4b^5} +, \text{\&c.}$$

Where the coefficients of the denominators arise from the multiplication into one another continuously of the terms of the A.P. 1, 2, 3, 4, 5, &c. And hence from a given logarithm the number corresponding to it can be found.

From these examples it may be seen how the bounds of Analysis are enlarged by means of infinite equations of this kind; indeed, by their aid the method extends, I had almost said, to all problems except Diophantine and similar questions. The method however

is not quite general except by means of certain further methods of forming series. For there are problems, in which one cannot arrive at infinite series by division, or by the extraction of roots, simple or affected. But there is no time to say what the procedure should be in such cases; nor to describe some other things which I had devised relating to the reduction of infinite into finite series, when the nature of the case permits. For I am writing somewhat briefly because these speculations have begun for some time past to be less interesting to me, so that indeed I have abstained from them now for almost five years.

Nevertheless I shall add one point; that after any problem is expressed in terms of an infinite equation, then various approximations for mechanical use can be found with hardly any labour; which when sought by other methods usually involve much labour and the expenditure of time.

[The letter concludes with an investigation of Huygen's approximate formula for the length of the arc of a circle:

If A be the chord of the arc, B the chord of half the arc, and r the radius, then if z be the length of the arc,

$$z = \frac{8B - A}{3} \text{ with an error less than } \frac{z^5}{7680r^4} \text{ in excess;}$$

and some further approximations are given relating to the circle, the ellipse, and the hyperbola].

Letter of 24 October 1676.

Newton to Oldenburg.

(To be communicated to Leibnitz.)

I could hardly say with how much pleasure I have read the letters of those illustrious men, Leibnitz and Tschirnhausius.

The method by which Leibnitz obtains convergent series is very elegant indeed, and would have shown the ability of the author if he had written nothing else. But remarks, most worthy of his reputation, which occur here and there in his letter, lead us to hope for the greatest things from him. The diversity of methods by which the same results are obtained is all the more attractive because three methods had already become known to me of arriving at series of this kind; so that I hardly expected that anything new could be communicated to us.

One of my methods I described in my former letter; now I add another; that, indeed, by which I was first led to these series; for I was led to them before I discovered the division (of fractions) and the extractions of roots which I now use. And in the explanation of the Theorem which was set forth at the beginning of my former letter is to be found the foundation which Leibnitz desires from me.

At the beginning of my mathematical studies, when I had fallen in with the works of our celebrated Wallis and came to consider

the series by the interpolation of which he brings out the areas of the circle and of the hyperbola, since in the series of curves whose basis or common axis is x and whose ordinates are :

$$(1-x^2)^{\frac{0}{2}},\ (1-x^2)^{\frac{1}{2}},\ (1-x^2)^{\frac{2}{2}},\ (1-x^2)^{\frac{3}{2}},\ (1-x^2)^{\frac{4}{2}},\ (1-x^2)^{\frac{5}{2}},\ \&c.,$$

if the areas of the alternate curves, which are

$$x,\ x-\frac{1}{3}x^3,\ x-\frac{2}{3}x^3+\frac{1}{5}x^5,\ x-\frac{3}{3}x^3+\frac{3}{5}x^5-\frac{1}{7}x^7,\ \&c.,$$

can be interpolated we should have the areas of the intermediate curves the first of which, $(1-x^2)^{\frac{1}{2}}$, is the circle; I noted for these interpolations that in every case the first term was x, that the second terms $\frac{0}{3}x^3$, $\frac{1}{3}x^3$, $\frac{2}{3}x^3$, $\frac{3}{3}x^3$, &c., were in A.P.; and accordingly that the two first terms of the series to be interpolated must be

$$x-\frac{1}{2}\cdot\frac{x^3}{3},\ x-\frac{3}{2}\frac{x^3}{3},\ x-\frac{5}{2}\frac{x^3}{3},\ \&c.$$

For the insertion of the remaining terms I considered that the denominators 1, 3, 5, 7, &c., were in A.P.; and so the numerical coefficients had to be investigated for the numerators only. But in the given alternate areas these were the figures which express the powers of the number 11; namely, 1; 1, 1; 1, 2, 1; 1, 3, 3, 1; 1, 4, 6, 4, 1, &c.

Then I set myself to enquire how in these groups of figures when the first two terms of a group were given the rest could be derived. And I found that assuming the second figure to be m the rest would be produced by the continuous multiplication of the terms,

$$\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times,\ \&c.$$

For example, let the second term m be $=4$; then the third term will be $4\times\frac{m-1}{2}$, that is 6; and the fourth $6\times\frac{m-2}{3}$, that is 4; and the fifth, $4\times\frac{m-3}{4}$, that is 1; and the sixth, $1\times\frac{m-4}{5}$, that is 0; with which the series in this case terminates.

I applied this rule therefore to obtain the intermediate series. And since for the circle the second term was $\frac{1}{2}\cdot\frac{x}{3}$, I put $m=\frac{1}{2}$ and the resulting terms were $\frac{1}{2}\times\frac{\frac{1}{2}-1}{2}$, or $-\frac{1}{8}$; $-\frac{1}{8}\times\frac{\frac{1}{2}-2}{3}$, or $\frac{1}{16}$; $\frac{1}{16}\times\frac{\frac{1}{2}-3}{4}$ or $-\frac{5}{128}$; and so on to infinity, whence I found that the

area sought for the segment of a circle was

$$x - \frac{1}{2}\frac{x^3}{3} - \frac{1}{8}\frac{x^5}{5} - \frac{1}{16}\frac{x^7}{7} - \frac{5}{128}\frac{x^9}{9}, \text{ \&c.}$$

By the same method also the intermediate areas of the remaining curves are obtained ; as also the area of the hyperbola and other alternate terms in this series,—

$$(1+x^2)^{\frac{0}{2}},\ (1+x^2)^{\frac{1}{2}},\ (1+x^2)^{\frac{2}{2}},\ (1+x^2)^{\frac{3}{2}}, \text{ \&c.}$$

And the method for obtaining intermediate series in other cases is the same, whether the intervals between two terms or more are wanting.

This was my first entrance into these speculations, which would have quite passed out of my memory had I not cast my eyes on certain memoranda a few weeks ago.

But when I had obtained these results, I soon began to consider that the terms $(1-x^2)^{\frac{0}{2}}$, $(1-x^2)^{\frac{2}{2}}$, $(1-x^2)^{\frac{4}{2}}$, $(1-x^2)^{\frac{6}{2}}$, could be interpolated in the same manner as the areas generated by them ; and for this nothing more was necessary than the omission of the denominators 1, 3, 5, 7, &c., in the terms expressing the areas. That is to say, the coefficients of the terms of the quantity to be interpolated, $(1-x^2)^{\frac{1}{2}}$, or $(1-x^2)^{\frac{3}{2}}$, or generally $(1-x^2)^m$, arise from the continuous multiplication of the terms of this series,

$$m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$$

So, for example,

$$(1-x^2)^{\frac{1}{2}} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6, \text{ \&c.}$$

$$(1-x^2)^{\frac{3}{2}} = 1 - \frac{3}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{16}x^6, \text{ \&c.}$$

In this way therefore the general reduction of radical expressions into infinite series became known to me by the rule which I set forth at the beginning of my former letter, before I discovered the extraction of roots. But when I had learned this, the other could not long be concealed. For in order that I might prove these operations I multiplied $1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6$, &c., into itself and the result was $(1-x^2)$, the remaining terms to infinity vanishing throughout the continuation of the series. And in the same way $1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 - \frac{5}{81}x^6$, &c., multiplied twice into itself produced $(1-x^2)$.

When I was sure of the demonstration of these conclusions I was led to try on the other hand whether these series which it proved to be roots of the quantity $(1-x^2)$ could not be extracted

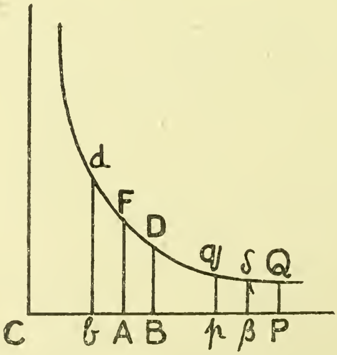
from it in the arithmetical manner; and the attempt was quite successful. The form of the operation in the case of the square root was this:

$$
\begin{array}{l}
1 - x^2\left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 -, \text{ \&c.}\right. \\
\underline{\quad 1 \quad} \\
0 - x^2 \\
\quad \underline{-x^2 + \frac{1}{4}x^4} \\
\qquad -\frac{1}{4}x^4 \\
\qquad \underline{-\frac{1}{4}x^4 + \frac{1}{8}x^6 + \frac{1}{64}x^8} \\
\qquad\qquad\quad -\frac{1}{8}x^6 - \frac{1}{64}x^8
\end{array}
$$

After realizing these consequences I entirely neglected the interpolation of series; and employed only these operations, as being more essentially fundamental. Nor was reduction by division concealed from me, a method very easy to use.

But I soon attacked the solution of affected equations, and obtained that also; from which, the ordinates, the segments of the areas and any other lines became known at once when the areas or arcs of the curves were given. For the regression to these results required nothing beyond the solution of the equations by which the areas or arcs were given in terms of the given lines.

At that time the increasing plague (which fell in the years 1665, 1666) compelled me to fly from this place and to turn my thoughts to other things. Nevertheless, immediately afterwards I produced a scheme, which I here subjoin, for the calculation of logarithms from the area of the hyperbola. Let dFD be a hyperbola whose centre is C, vertex F, and intercepted square CAFE$=1$.



[The completion of the square, and the letter E do not occur in Newton's diagram.]

In AC take AB and Ab on one side and the other, $=\frac{1}{10}$ or 0·1; and having erected the perpendiculars BD, bd, with their extremities on the hyperbola, the semi-sum of the spaces, AD and Ad

$$=0\cdot1+\frac{0\cdot001}{3}+\frac{0\cdot00001}{5}+\frac{0\cdot0000001}{7}, \text{ \&c.};$$

and their semi-difference

$$=\frac{0\cdot01}{2}+\frac{0\cdot0001}{4}+\frac{0\cdot000001}{6}+\frac{0\cdot00000001}{8}, \text{ \&c.}$$

And the calculation of these terms gives the following results:

0·100, 000, 000, 000, 0	0·005, 000, 000, 000, 0
333, 333, 333, 3	25, 000, 000, 0
2, 000, 000, 0	166, 666, 6
14, 285, 7	1, 250, 0
111, 1	10, 0
9	1
0·100, 335, 347, 731, 0	0·005, 025, 167, 926, 7

The sum of these 0·105, 360, 515, 657, 7, is Ad; and their difference 0·095, 310, 179, 804, 3, is AD. And in the same way AB and Ab being taken on one side and the other $=0\cdot2$, Ad will be found $=0\cdot223$, 143, 551, 314, 2; and AD $=0\cdot182$, 321, 556, 793, 9. After the hyperbolic logarithms of the four decimal numbers 0·8, 0·9, 1·1, and 1·2, have been obtained in this way; since $\frac{1\cdot2}{0\cdot8}\times\frac{1\cdot2}{0\cdot9}=2$; and 0·8 and 0·9 are less than unity, by adding their logarithms to twice the logarithm of 1·2, you get 0·693, 147, 180, 559, 7 as the hyperbolic logarithm of the number 2. Since $\frac{2\times2\times2}{0\cdot8}=10$, by adding the logarithm of 0·8 to three times the logarithm of 2 you will get 2·302, 585, 092, 993, 3 the logarithm of the number 10; and then by addition the logarithms of the numbers 9 and 11 are at once obtained, and thus the logarithms of all these prime numbers, 2, 3, 5, 11, are ready at hand. And further, simply by the depression of the numbers used in the decimal calculation detailed above, and by addition, the logarithms of the decimal numbers 0·98, 0·99, 1·01, 1·02, are obtained; as also of the decimal numbers 0·998, 0·999, 1·001, 1·002; and thence by addition and subtraction, the logarithms of the prime numbers 7, 13, 17, 37, &c., are obtained. Which, together with those found above, when divided by the logarithm of the number 10 give the true logarithms for insertion in a table. But these I afterwards obtained more exactly.

[*Footnote by Editors of "Commercium Epistolicum." Vide* "Geometriam Analyticam", Cap. ix, pp. 36–48.]

I am ashamed to say to what a number of decimal places I carried these calculations being then at leisure. For, indeed, I took too much pleasure at that time in these investigations. But when the ingenious Logarithmotechnia of Nicholas Mercator appeared (whom I suppose to have been the first discoverer of his methods), I began to take less interest in them; imagining that he was acquainted with the extraction of roots as well as the division of fractions; or that others, once the method of division had been brought to light, would soon discover the rest before I was of mature age for writing.

[*Footnote by Editors of "Commercium Epistolicum."*—Earlier mathematicians discovered this theorem, that the sum of the terms of a Geometrical Progression proceeding to infinity, is in the same ratio to the first and greatest of the terms, as this term is to the difference between the first and second terms. This is proved arithmetically by multiplying the extremes and means of the ratio. Wallis proved it by dividing the product of the means by the last term of the ratio. See the Opus Arithmeticum of Wallis, published in 1657, Cap. xxxiii, § 36. By using Wallis' method of division, Mercator proved and extended the quadrature of the hyperbola, previously discovered by Brounker. And Gregory proved the same geometrically. But none of these discovered the general method of finding the areas of curves. Mercator nowhere claimed this. Gregory who was admittedly a man of the greatest ability, and who had his attention drawn to the subject by the letters of Collins, at length and with difficulty found a method of this kind. Newton found it by the interpolation of series, and afterwards by the divisions and extractions of roots as being more familiar.]

At the time when that book appeared, a compendium of the method of these series was communicated to Collins by my friend Barrow, the Professor of Mathematics at Cambridge. In that compendium I had shown how the areas and lengths of all curves and the surfaces and volumes of solid bodies could be found from their given ordinates, and *vice versâ*, how the ordinates could be found if the areas, &c., were given; also I had illustrated the method there given by various series.

A regular correspondence having thereupon sprung up between us, Collins, who was devoted to the promotion of mathematical knowledge, did not cease to urge that I should make these results public. And five years ago (1671) when on the advice of my friends I was planning the publication of a treatise on the refraction of light and on colours which I then had in readiness, I began again to think of these series and wrote a treatise on them with the intention of publishing both together.

But, arising out of the catadioptric telescope, after I had written you a letter in which I briefly explained my ideas on the subject of light; something unexpected brought it about that I felt it to be of importance to me to write to you hastily about the printing of that letter. And the number of questions that immediately arose, through letters of various people crammed with objections, &c., deterred me entirely from my plan; and had the result that I accused myself of imprudence, because by grasping at the shadow I had lost my peace, a thing of real substance.

About that time James Gregory, from a single series of mine which Collins had sent him, arrived at the same method, after, as

he said in a letter to Collins, a great deal of consideration ; and he left a treatise on the subject which we hope will be published by his friends. For, from the ability he possessed he could not but add much new matter of his own, and it is of importance in the interests of mathematical knowledge that it should not be lost. Moreover, I had not completely finished my treatise when I desisted from my plan ; nor to this day has my mind returned again to the things that were left to be added. That part indeed was wanting in which I had proposed to explain the method of solving problems which could not be reduced to quadratures ; granted that I might have done something towards the foundation of it. Moreover, infinite series did not occupy much space in that treatise.

* * * * *

Although many things remain to be investigated about methods of approximation, and about different kinds of series which may serve for that purpose ; nevertheless I should hardly hope with Tschirnhausius that simpler or more general methods of reducing quantities to the kind of series in question can be given than the divisions and the extractions of roots, which Leibnitz and myself use ; at any rate not more general, because for quadrature and rectification of curves and similar questions, no series can be given depending on these simple algebraical terms (involving only one indefinite quantity) which it is not possible to obtain by this method.

For the number of convergent series for the determination of the same quantity cannot be greater than the number of indefinite quantities from the powers of which the series are produced, and I am acquainted with methods of obtaining a series from any indefinite quantity that may be employed ; and I believe that Leibnitz also has that in his power.

For although by my method there is a free choice, for the construction of the series, of any indefinite quantity on which the question depends, and the method which he has communicated to us seems to be adapted for the choice of such indefinite quantities as can conveniently be reduced to fractions, which, by division only, produce infinite series ; nevertheless any other indefinite quantities whatever can be employed for the construction of series by means of the method used for the solution of affected equations, provided they are solved in appropriate terms, that is by constructing the series only from terms which are involved in the question.

Moreover, I do not see why it should be said that by using these divisions and resolutions, problems are solved by accident, since these operations have the same relation to this kind of algebra as the common operations of arithmetic to ordinary algebra.

But as regards simplicity of method ; I would not have fractions and radicals resolved invariably into infinite series without previous reduction ; when complicated quantities occur, all kinds of reductions are to be tried ; whether by increasing, diminishing, multiplying or dividing the indefinite quantities ; or by Leibnitz' method of transmutation ; or by any other method which may happen to

fit the case ; and then resolution into series will be suitably employed by division and extraction.

Moreover, efforts should specially be made to reduce the denominators of fractions, and quantities under the radical sign, to the fewest and least complicated terms possible ; and to such also as are most rapidly expanded into convergent series, although the roots may be neither converted into fractions nor depressed. For by the rule given at the beginning of the earlier letter, the extraction of the highest roots is as simple and easy as the extraction of the square root or division ; and series which result from division are usually the least convergent of all.

Hitherto I have spoken of series involving only one indefinite quantity. But series can also by the method investigated be constructed at pleasure from two or from more indefinite quantities. Moreover, by the aid of the same method series can be formed for all curves, of a character similar to the series given by Gregory for the circle and hyperbola ; that is series of which the final term gives the area sought. But I would not willingly undertake this calculation.

Finally, series can be derived from complex expressions by the same method. As for example, if $\sqrt{a^2 - ax + \frac{x^3}{a}}$ be the ordinate of any curve, I put $a^2 - ax = z^2$, and the extraction of the square root of the binomial, $z^2 + \frac{x^3}{a}$, will produce $z + \frac{x^3}{2az} - \frac{x^6}{8a^2z^3}$, &c.

All the terms of this series can be quadrated by the theorem already described. But I attach little importance to this method because when simple series are not obtainable with sufficient ease, I have another method not yet published by which the problem is easily dealt with. It is based upon a convenient, ready and general solution of this problem, *To describe a geometrical curve which shall pass through any given points.*

Euclid has shown how to describe a circle through three given points. Also a conic section can be described through five given points and a curve of three dimensions through seven given points ; so that I have in my power a description of all the curves of that order which are determined by seven points only. These are done at once by geometrical methods, without any calculation. But the above problem is of another kind, and although it may seem to be intractable at first sight, it is nevertheless quite the contrary ; perhaps indeed it is one of the prettiest problems that I can ever hope to solve.

* * * * *

Nor when he [*i.e.*, Leibnitz] divides this series

$$\frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} +, \text{\&c.},$$

does he seem to have observed my general method of using letters in the place of quantities affected with their signs + and −. For

since the hyperbolic area AD, here signified by z, is positive or negative according as it lies on one or the other side of the ordinate AF; if that area given in numbers is l, and l is substituted for z in the series, the result will be either $\frac{l}{b}+\frac{l^2}{2ab^2}+\frac{l^3}{6a^2b^3}+\frac{l^4}{24a^3b^4}+$, &c., or $-\frac{l}{b}+\frac{l^2}{2ab^2}-\frac{l^3}{6a^2b^3}+\frac{l^4}{24a^3b^4}$, &c., according as l is positive or negative. This being understood, if $a=1=b$, and l stands for the hyperbolic logarithm, the number corresponding to it will be $1+\frac{l}{1}+\frac{l^2}{2}+\frac{l^3}{6}+\frac{l^4}{24}$, &c., if l be positive; and $1-\frac{l}{1}+\frac{l^2}{2}-\frac{l^3}{6}+\frac{l^4}{24}$, &c., if l be negative. In this way I avoid the multiplication of theorems, which otherwise would increase to an extraordinary degree. For to take an example, that one theorem which I gave above [omitted from the present version] for the quadrature of curves would be resolved into 32, if it were multiplied in accordance with the variations of sign.

Moreover, I do not yet understand what my eminent friend says about finding a number, greater than unity, from its hyperbolic logarithm by the use of the series

$$\frac{l}{1}-\frac{l^2}{1\times 2}+\frac{l^3}{1\times 2\times 3}-\frac{l^4}{1\times 2\times 3\times 4}+, \text{\&c.},$$

rather than by the use of the series

$$\frac{l}{1}+\frac{l^2}{1\times 2}+\frac{l^3}{1\times 2\times 3}+\frac{l^4}{1\times 2\times 3\times 4}+$$

For if one term more be added to the latter series than to the former, the latter will give a better approximation. And certainly it is less laborious to calculate one or two figures of this additional term than to divide unity by the number extended to many decimal places, derived from the hyperbolic logarithm, in order that the required number greater than unity may then be obtained. Therefore let either series, if it be right to speak of two, be employed for its appropriate work. Nevertheless the series

$$\frac{l}{1}+\frac{l^3}{1\times 2\times 3}+\frac{l^5}{1\times 2\times 3\times 4\times 5}, \text{\&c.},$$

depending on half the terms can be best employed, since this will give the semi-difference of two numbers; from which, and from the given rectangle either number is given. So also from the series

$$1-\frac{l^2}{1\times 2}+\frac{l^4}{1\times 2\times 3\times 4}, \text{\&c.},$$

the semi-sum of the numbers is given, and thence the numbers themselves. From which arises a relation between the two series so that when one is given the other is found.

From the following simple process which depends on such series, you will readily agree that the construction of logarithms need not be attempted in any other way.

By the method previously explained the hyperbolic logarithms of the numbers 10, ·98, ·99, 1·01, 1·02 are investigated, which would occupy an hour or so. Then, dividing the logarithms of the four last numbers by the logarithms of the number 10, and adding the index 2, the true logarithms of the numbers 98, 99, 100, 101, 102 are obtained for entry in a table. These are to be interpolated by intervals of one-tenth (*Hi per dena intervalla interpolandi sunt*) and the logarithms of all numbers between 980 and 1,020 will be found; the numbers between 980 and 1,000 being again interpolated by intervals of one-tenth, the table will so far be constructed. Then from these are to be collected the logarithms of all the primes, less than 100, and their multiples; for which nothing but addition and subtraction are required. Thus:

$$\sqrt[10]{\frac{9984 \times 1020}{9945}} = 2; \quad \sqrt[4]{\frac{8 \times 9963}{984}} = 3; \quad \frac{10}{2} = 5; \quad \sqrt{\frac{98}{2}} = 7; \quad \frac{99}{9} = 11;$$

$$\frac{1001}{7 \times 11} = 13; \quad \frac{102}{6} = 17; \quad \frac{988}{4 \times 13} = 19; \quad \frac{9936}{16 \times 27} = 23; \quad \frac{986}{2 \times 17} = 29;$$

$$\frac{992}{32} = 31; \quad \frac{999}{27} = 37; \quad \frac{984}{24} = 41; \quad \frac{989}{23} = 43; \quad \frac{987}{21} = 47;$$

$$\frac{9911}{11 \times 17} = 53; \quad \frac{9971}{13 \times 13} = 59; \quad \frac{9882}{2 \times 81} = 61; \quad \frac{9949}{3 \times 49} = 67; \quad \frac{994}{14} = 71;$$

$$\frac{9928}{8 \times 17} = 73; \quad \frac{9954}{7 \times 18} = 79; \quad \frac{996}{12} = 83; \quad \frac{9968}{7 \times 16} = 89; \quad \frac{9894}{6 \times 17} = 97.$$

And so, having found the logarithms of all numbers less than 100, it remains only to interpolate these also once and again by intervals of one-tenth.

* * * * *

[After a section on the construction of Trigonometrical Tables the following paragraph occurs]:

What has been said about Tables of this kind can be applied to others where Geometrical considerations have no place. Moreover, it is sufficient by means of these series to calculate 30 or 20, or even fewer terms at suitable distances apart, since the intermediate terms are easily inserted by a method which I had almost decided to describe here for the use of computers. But I pass on to other matters.

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Notes on some recent developments of Pension Problems in America. *By* W. J. H. Whittall, F.I.A., F.A.S.

The Carnegie Foundation (*continued from p.* 117).

SUMMARIZING the policy of the Carnegie Foundation it is seen that it covers the following fundamental principles : (i) that contributions by the employed are not only necessary, but desirable in themselves ; (ii) that death benefits are desirable ; (iii) that the proper method of procedure is by contractual methods (*i.e.*, by what we should call the money purchase method), based on contributions provided in advance ; (iv) that the reward for past service must be separately provided for ; and (v) that disablement must be regarded as a separate problem. In all these respects the conclusions are exactly the same as those arrived at independently, after a long argument, in the Report on Hospital Officers' Pensions, recently published by King Edward's Fund.

Of the points I have enumerated the first appears to me to be the most important. The Carnegie Foundation appears to have faced this question boldly and have actually converted the Teachers as a class to the view that contributions are not only necessary, but desirable in themselves. The Joint Commission, which included representatives of the University professors, laid down the fundamental principle, as a "primary obligation of the existing social order," that the individual should provide protection for himself and for his family. To this they added that the employer must assume some part of the obligation. The Carnegie Trustees go further, and assert (i) that no system of free pensions can be devised which will not in the end affect the teacher's pay ; (ii) that a contributory system is the only one which society can permanently support, and (iii) that, according to the "experience of the world," any group in the body politic receive their best service from society if they are assisted to economic independence.

The amended rules of the Carnegie Foundation have recently reached me. Its method of reducing its own commitments for the existing staffs is to increase the age of retirement to 70, with the option of smaller allowances at earlier ages. This solution of their difficulty will of course leave the beneficiaries free to supplement their pensions on their own account by contributing to the new Association for annuities in respect of their

future service. As regards all new appointments the Foundation will guarantee an average return of $4\frac{1}{2}$ per-cent on payments made to the Teachers' Association for deferred annuities, and definitely undertakes the provision of the disability allowances. On disability, the teacher will assign his annuity policy to the Foundation and will receive an immediate annuity of two-thirds of the amount of the ultimate benefit therein assured. This undertaking is limited to the next 20 years, after which time the Trustees think that experience will enable the disability risk to be measured actuarially and contracted for by the Teachers' Association. No suggestion is made that the Death Benefit, if any, assured in the Association should be assigned to the Foundation or that this should in any way be co-ordinated with the other assets to provide disability benefit in the manner suggested by Mr. F. L. Collins in the Report on Hospital Officers to King Edward's Fund. On the contrary, there is to be a special provision in all the Insurance Contracts of the Teachers' Association that the policy shall continue in full force without further payment of premiums in the event of complete disability before age 65. It would thus seem to be the intention, in that event, to preserve for the assured the full benefit of such insurance as he may have effected. The British Offices who make a speciality of catering for pension schemes might well consider the possibility of providing for this contingency.

The detailed "Handbook of the Teachers' Insurance and Annuity Association of America" gives effect to the scheme on the lines already indicated.

It is perhaps to be regretted that the Life Insurance Companies of the United States cannot have the opportunity of contributing towards the solution of these pension problems. Our experience here is that such companies, with their widespread interests and established businesses, are in a better position to enter into annuity contracts than any new organization can be.

As the Carnegie Foundation supplementary benefits are limited to holders of and are based on the amounts of the annuity contracts, the joint contributions will be spent mostly if not wholly on that form of contract. It may be worth while to give a few specimens of the rates quoted for annuities and insurances.

Rates of Monthly Annuity granted by the Teachers' Association.

Age	Deferred to Age 65: Per Monthly Premium of $10, Male	Deferred to Age 65: Per Monthly Premium of $10, Female	Deferred to Age 65: Per Single Premium of $100, Male	Deferred to Age 65: Per Single Premium of $100, Female	Age	Immediate: Per Single Premium of $1,000, Male	Immediate: Per Single Premium of $1,000, Female
25	$109.81	$96.27	$4.53	$3.97	50	$6.26	$5.66
30	85.11	74.62	3.72	3.26	60	8.06	7.15
40	48.13	42.19	2.51	2.20	65	9.52	8.33
50	23·14	20.29	1.70	1.49	70	11.59	10.00

Annual Premiums for $1,000 *Insurances.*

Age	Term Insurance: Ten-Year	Term Insurance: Expiring at Age 60	Term Insurance: Expiring at Age 65	Term Insurance: Expiring at Age 70	Whole Life	Endowment Insurance at 65
21	$7.93	$9.81	$10.55	$11.43	$13.91	$16.50
30	8.69	11.07	12.20	13.64	17.40	21.91
40	10.78	13.34	15.31	17.60	23.88	33.32
50	...	17.59	20.82	24.71	35.75	61.02

To meet the case of members who withdraw from the teaching profession, the rates of premium reserved in the contracts will be higher than the foregoing, a reduction of 10 per cent being allowed so long as the assured remains in the profession. The full premiums payable on withdrawal from educational employment would therefore be the quoted rates increased by one-ninth. It seems that a provision of this kind is quite fair if it means, as I assume, that a withdrawing member will in all cases have the option of maintaining his annuity and other contracts for his own benefit, together with the full benefit of the cash surrender values and other standard provisions of the New York Insurance Law.

On this question of the "vested rights" principle, to which I have drawn attention on another page and which seems to be essentially the proper complement and corollary of the contributory method, I cannot find that the Carnegie Foundation has laid down a definite ruling, but the Handbook of the Teachers' Association clearly contemplates that the principle of the vested right accruing from year to year will be recognized

in practice. "Teachers who have followed", it is remarked, "the discussions of pensions during recent years will understand that the contribution made by a college or university "to a teacher's annuity will inevitably in the course of time "be considered as part of his salary. This result must always "follow in any such arrangement between two parties who "have to each other the relation of employer and employee. "Nevertheless, even when this stage has been reached, it is still "of great advantage to the teacher to have his college advance "this payment, even though it be in the nature of deferred salary. "Not only is it an advance payment, but its full value can be "secured to the teacher whether he lives out his term of activity "and makes use of the annuity, whether he changes his occupation, or whether he dies prematurely."

It would seem that in this question of vested rights, the Trustees of the Foundation have taken up very much the position that the Committee on Hospital Officers Pensions recommended King Edward's Fund to adopt, namely, that while not going the length of laying down the principle as an actual condition of its support, it should cast its influence as far as possible on the side of establishing it.*

THE STATE OF ILLINOIS.

The State of Illinois appointed in 1916 a Commission (with an Actuary, Mr. Donald F. Campbell, as its Secretary) to investigate all its pension laws, and there was found to be not only want of foresight and ignorance as to the ultimate cost of the liabilities, but a quite chaotic variety of conditions for determining the pensions themselves. In order to determine the liabilities the principal funds were thoroughly investigated, Life and Service Tables being prepared. On four of the principal valuations the total present deficiency appears to be about £11,000,000. As to the future, the Commission proposed that the pensions should be provided in advance, that the employees should contribute, and that there should be death benefits for dependants. A minimum compulsory pension is suggested at fifty-five, with a voluntary "sur-pension" proportionate to any increase of salary. The cost of existing staffs is to be met mainly by the State and charged over forty years.

* Since writing this I learn that the Deferred Annuity Form of Contract definitely assures the whole benefit to the member, whatever proportion (even if the whole) of the premium be paid by the Institution.

The Report makes the following statement in regard to our Civil Service Pension system :

> After a hundred years of experience with Civil Service pensions in England, the pensions are regarded very generally as deferred pay. . . . . . It was in the law of 1909 that the principle was recognized that even with a non-contributory system the employee who withdraws from the Service should receive certain benefits. . . . . . In other words, even a non-contributory system of pensions has in England become virtually contributory, because wages and salaries are held at a lower level on account of the pension prospects.

It seems clear that this statement is inaccurate in regard to withdrawal benefits. It is to be regretted that the language of our Act of 1909, presumably Sec. 1 (2), should have misled an American student on such an important point, but the practice of the Treasury is well known. Throughout our Civil Service no benefit whatever is paid on voluntary withdrawal earlier than the " retirement " recognized by the Acts. For American practice I quote the following precise and important statement in the Report :

> Without exception, Pension Acts in the United States* that do not provide for contributions from salaries of employees make no provision for any payment to employees upon separation from active service on account of resignation or dismissal ; and in those cases where such contributions are required, the payment never exceeds the amount contributed by the employee.

New York City Pensions Commission.

The Corporation of New York City has recently made what is probably the most thorough investigation into pension liabilities ever undertaken by a public authority. A Pensions Commission was appointed, of which Mr. George W. Perkins, known to many of us as a former Vice-President of the New York Life Insurance Company, was Chairman ; and a special Actuarial staff was engaged with Mr. G. B. Buck, F.A.S., as its chief. The results are contained in several volumes packed with statistical and actuarial information. The first part of the Report was presented in February 1916, consisting of a preliminary statistical account of the nine systems covering the various city services. This was followed by an actuarial investigation and valuation of them ; and finally a new scheme was proposed which in turn required a large amount of further

* Over 200 are cited !

actuarial work to estimate the cost of it. For the following notes I have consulted the original reports.

The New York City pension systems originated with the employees. The Government was indifferent. Thus in 1857 the Police Force procured special legislation giving them pensions. In 1871 the Fire Brigade secured pensions, and in 1894 the Teachers. Other funds followed, including the "Grady" Fund, and (in 1911) one for the Street Cleaning Department. In 1911 also the Grady Fund was extended to include all employees not covered by the other Funds. Finally, the system consisted of the foregoing five funds and four smaller funds. The five large funds include 75,359 employees out of the total of 77,310, and for brevity I will deal only with these in giving details. The statistical position was as follows in 1914:

Fund	Founded	Employees 1914	Pensioners 1914	Total Receipts since Establishment from		Balance Dec. 31 1914
				Contributions	City Funds and Interest, &c.	
				£	£	£
Police ...	1857	10,708	4,234	917,030	5,521,689	146
Fire Dept. ...	1871	5,004	1,686	...	2,816,050	170,324
Teachers ...	1894	21,317	1,549	444,525	1,817,104	181,321
Grady ...	1906	32,856	166	...	34,261	None
Street Cleaning	1911	5,474	484	85,743	175,643	201,260
All Funds ...	...	77,310	8,232	1,469,812	10,535,843	622,379

The 8,232 pensioners consisted of 5,779 employees with an average pension of £152, and 2,453 dependants with an average of £61. The total salary list of all the funds as at June 1914, was £19,590,182.

In the Fire Department and Teachers Funds, Superannuation of half final pay is granted after 20 and 30 years' service respectively, irrespective of age. A similar scale applies to the Police after 25 years and the Street Cleaning Department after 20 years, but with minimum ages of 55 and 60 respectively. In the Grady Fund there is no superannuation provision; but on disability after 30 years' service half final pay is given. Disability is also provided for in the other four Funds on a generous scale, which is frequently half final pay after 20 years' service, and sometimes after 10 years. There are also benefits for dependants in the case of the Police, Fire and Street Cleaning Funds.

For these benefits the Police contribute 2 per-cent of salary, the Teachers 1 per-cent, and the Street Cleaners 3 per-cent. In the other two Funds there are no contributions. Many of the Funds enjoy special appropriations of excise monies and other special items of the City's Income, which therefore do not pass through the City Accounts and thus reduce the apparent ultimate cost to the City by way of direct contributions. In the above table this differentiation is disregarded, as the whole of the City's contributions are public money. As will be judged from the small balances in hand, it has been usual to pay pensions out of the funds, as in the case of our own Police Pensions, regardless of any actuarial valuations; and it is the recent increase in direct contributions demanded from the City which has led to the investigation. The Pensions Commission found that, "due to their fitful and unsystematic development, the existing pension laws present a tangled mass of conflicting provisions."

It would serve no practical purpose to follow the Commission in their painstaking exposition of the inequalities and inconsistencies which they found. In their summary they remark

(i) that employees in different branches get varying benefits not justified by differences in their work;

(ii) that the employees in each fund get varying benefits through the operation of crude flat pension rates and contributions;

(iii) that about 37 per-cent of the municipal service could retire after 20 to 30 years service irrespective of physical condition;

(iv) that on the other hand about 43 per-cent of the service had no superannuation but only disability allowance to look forward to, and there were already 3,323 employees above age 60 who could not retire even if inefficient;

(v) that the frequent employment of pensioners at fresh salaries led to absurd results; and generally that the regulations as to disability, as to dependants and as to contributions were all inconsistent with one another and based on no general principles.

As regards the financial provisions of the existing system the Commission find

(i) that the schemes were all launched without actuarial advice or proper financial provision for their cost;

(ii) that the City's support was disguised by the allotment of indirect sources of revenue ;

(iii) that the City was directly responsible for four funds, of which the two important ones, the Police and Fire Department, were likely to develope respectively a pensions expenditure of 35 per-cent and 45 per-cent of the pay roll ; and

(iv) that five of the funds, including the important Teachers Fund, are limited to definite sources of income, and are now, or soon will be, bankrupt.

The Commission also reported that owing to inadequate departmental records a costly census and much delay had been necessitated before the actuarial staff could construct the mortality and service tables needed for calculations of cost.

Meanwhile, the Commission proceeded to arrive at certain preliminary conclusions as to the proper fundamental principles of a new system. They did not attempt to reconcile the illogical features of the existing system, which, they say, would have "collapsed under the weight of its own absurdities, but for the "lack of general knowledge of the details, and the oblique "methods of financing by which its cost in the past had been "hidden" ; but before determining on any new financial basis they made a broad survey of existing pension systems in operation, both in the United States and abroad. They found "that "the development of pension measures as a result of an experience "of over a hundred years is in the direction of equal division of "cost between the employer and the employee, and that this "tendency applies equally to systems for public employees and "for industrial workers."

Part II of the Report comprising the Actuarial Investigation was published in May 1916. The Actuarial Society of America had appointed by request an honorary advisory Committee of Actuaries to assist the Commission, consisting of Messrs. William A. Hutcheson (Chairman), Robert Henderson and Henry Moir ; and the work of Mr. Buck and his actuarial staff was carried out with the help and approval of these gentlemen. A six-year period, from 1908 to 1914, was selected, for which particulars of each employee were obtained on an elaborate form of card, and the ensuing statistical analysis, the Commission say, "required more than two years' work, and is "unparalleled in its scope by any similar study of a body of "employees ever made anywhere in the world."

The ultimate financial result, to which the whole Investigation led up, was to show, on the basis of existing laws and past experience, a present liability for existing staffs of £43,104,083. Deducting funds in hand, £769,931, and the value of future contributions, £1,779,038, there is shown a deficit of £40,555,114. The City, the Commission remark, had blindly embarked on a vast financial programme with no thought of the cost involved, and even if the inequalities of the existing schemes had not called for reform, the financial burden demanded a revision which would relieve taxpayers in part and provide for accumulation of reserves.

Part III of the Report, to which the whole of the foregoing investigation may be said to be ancillary, is dated February 1918, and contains the draft of the proposed new scheme followed by a full account of the actuarial calculations of its cost.

It is satisfactory to find that the new scheme begins by differentiating clearly between past service and future service. The futility of attempting to combine the two is amply shown in the history of our own Municipal schemes as given by Messrs. Manly and Ackland (*see J.I.A.*, vol. xlvi, p. 327). For the recompense of past service the City will assume the whole liability in the case of all employees who elect to enter the new scheme. Such entrance, optional for the time being, is subject to payment of contributions as at their present ages.

Commencing then with all future service the scheme provides service pensions on the salary percentage method, the basis being the average salary for the final ten years of service. The employee will contribute the cost of one half of this pension on a returnable basis; the City will provide the cost of the other half, but non-returnable. The City will further contribute

(i) the whole cost of disablement pensions, as well as pensions to widows and children, due to hazards arising in performance of duty;

(ii) the whole cost of certain benefits on death from ordinary causes; and

(iii) the greater part of the cost of pensions on disablement arising from ordinary causes after 10 years' service.

The scale of these proposed benefits varies in different branches of the service. Thus for service pensions the street cleaners are to receive 1/60th for each year of service at minimum age 55, the police, firemen and labourers 1/66th at age 58, the

mechanics 1/68th at age 59, the clerks and all administrative officers 1/70th at age 60, and teachers 1/74th at age 62. Retirement is to be compulsory in all cases at 70. The past service of existing officers entering the scheme is arranged for by the City undertaking to provide the whole "service fraction", instead of one-half, in respect of every prior year's service.

On disablement arising at any time in actual performance of duty the City will pay a pension of three-fourths of the final average salary, and the officer will retain the right to his own contributions. On disablement after 10 years' service from other causes, the pension will be nine-tenths of the corresponding service pension, with a minimum of 25 per-cent of final average salary. In this case the officer's own contributions are used *pro tanto* and the City provides the balance.

On death in actual performance of duty the City will pay to widow, children or dependent parents a pension of one-half of the final average salary, in addition to the returnable contributions. On death from other causes the benefit is a lump sum of one-half of the last year's salary, in addition to the contributions. On withdrawal from any cause the officer's own contributions are returnable with interest. Finally, the service pensions may be varied (as in the case of the Carnegie scheme) to provide a capital sum at death or to introduce a second life; and it is contemplated that all City contributions will be provided in advance, that proper reserves will be maintained on an actuarial basis, and that the management will be in a Board of Trustees on which each group of employees will be represented.

It may be thought I am lingering too long over details, but it appears to me that the great and pressing need at present in the pension problem is to settle fundamental questions of principle. The New York investigation probably represents the most thoroughgoing enquiry that has been made hitherto not only into objective facts but also into the general principles that should be adopted for future guidance. It is worth noting therefore the following general principles proposed for adoption:

(i) After the fullest consideration of alternatives it was decided that New York (like London) should exact contributions from its employees. Employer and employee are each to contribute strictly in advance a moiety of the estimated cost of the service pension. But while the employee contributes on a returnable basis the City retains its profit on withdrawals towards

other benefits. Thus there will be fostered a "sense "of mutual responsibility for the provision of benefits "which will be equitable and advantageous to both "parties."

(ii) Benefits should be given if possible with equal fairness to all. Hence, the services being in complex variety, the City charges itself with the whole cost of special hazards where such exist. Thus arise the special benefits where death or disablement occur in performance of duty—benefits which, as I understand, replace the benefits under the Workmen's Compensation Acts for which ordinary employers would be liable; and the frank recognition of the employee's claim to his own contributions in addition.

(iii) In adjusting scales of contributions and benefits the "coat was cut according to the cloth." The general aim was to provide for an entrant of 25 a service pension of one-half of the final average salary. It was then decided that the employee's contribution at that age on a savings bank basis should be as nearly as possible 4 per-cent of salary; and the retirement ages according to the various occupational experiences, with the resulting "service fractions", were thus arrived at as the conclusion of the process.

(iv) The City frankly accepts the additional cost for existing staffs. For example, a clerk can retire at 60, his service fraction being 1/70th for each year, of which 1/140th is provided by his own contributions. In the case of an existing officer electing to enter the scheme the City provides, in addition to its own normal moiety, the clerk's 1/140th for each year of prior service.

(v) Disablement from ordinary causes is not viewed too sympathetically, but I suspect that abuses under the existing system account in part for this. It is admitted that from the standpoint of the employee "disability is probably the danger which is viewed with "the gravest concern"; but to avoid offering temptation to retire early, and seek other employment, it was decided to limit the pensions to nine-tenths of the corresponding scale for service pensions. It is, however, contemplated that special additional grants might be

made by the City in meritorious cases and that the general scale of benefit might need to be broadened later.

On the foregoing basis rates of contribution are given for at least eleven different occupational groups. I select four of the groups for quotation.

Salary Percentages required.

Age at Entry	Employee's Contribution				City Contribution			
	Police	Fire Depart.	Teachers (Women)	Street Cleaning	Police	Fire Depart.	Teachers (Women)	Street Cleaning
20	4·01	4·35	4·08	3·57	4·53	3·93	2·77	2·14
25	4·12	4·47	4·12	3·96	5·10	4·41	3·15	2·85
30	4·33	4·69	4·22	4·41	5·58	4·86	3·63	3·53
35	4·67	4·94	4·43	4·91	5·53	5·13	3·98	4·23
40	5·10	5·24	4·77	5·46	5·48	5·60	4·35	5·31

We are finally presented with the results of a complete valuation of each benefit for each Fund on the assumption that every existing employee elects to come under the new scheme. The total liability for the whole service is as follows:

Existing Pensioners and Dependants	£10,590,342
Pensions in respect of Future Service (*less* Employees' half)	5,010,260
Disability: Ordinary (*less* Employees' share) ...	4,713,281
Do. Accident	490,661
Death Benefits: Ordinary	1,292,746
Do. Accident	793,979
Cost of Past Service	9,125,208
Grand Total	32,016,477
Less Cash in Existing Funds	769,931
Deficiency, to be provided by City ...	£31,246,546

It remains to be seen in what way the City will provide the cost of benefits for past service in the case of existing staffs. The London County Council in 1914 consolidated all its pension liabilities then outstanding and decided to liquidate the amount (£1,181,398) by an annuity spread over 50 years.

Note on a fundamental question underlying the Pension Problem; with a reference to the School Teachers' Superannuation Act, 1918. *By* W. J. H. Whittall, F.I.A., F.A.S.

It may be worth while, in view of current pension developments in America and elsewhere, to take stock briefly of the present position of the Pension Problem in this country; I mean of its broad aspects. I exclude from consideration the enormous liabilities the State has assumed for naval and military pensions. They have received insufficient attention from actuaries and statists, and the public has little conception of the burden it has assumed or how it is to be provided for when the existing wild orgy of expenditure shall cease.

These pensions, however, are incumbent, and will gradually decrease. Subject to the public insisting that they be provided for out of current taxation, and that no attempt be made to pay them out of borrowed money, they will settle themselves in time. For the purpose of this note I shall speak only of the civilian pensions of the future.

If we follow existing preconceptions we shall begin by differentiating between State pensions, having unlimited public money behind them, and all others. These others may be subdivided into categories,

(i) Where the employers have ample funds available for contributions or guarantees, including local authorities with rating powers, and

(ii) Where the beneficiaries must seek security for themselves.

Now the first fact which emerges from a review of the position in America is that a great heartsearching is taking place there as to the basic propriety of postponing State pension liabilities to be met by future generations. The growing tendency there seems to be to regard each State like any other employer, and to call on it to face its pension liabilities in advance. The questions involved in the proper determination of this apparently simple but fundamental question are so numerous and involved, and the sums concerned so great and so constantly growing, that it calls for particular and separate investigation. The Institute might well initiate this by devoting an evening specially to its discussion. A strict provision in advance for all State pensions might conceivably be the key to another pressing problem, namely, how to reduce the National Debt.

After the State pensions there is a large category of well secured benefits enjoyed by employees of municipalities or other statutory bodies with local rates to fall back on, and of various wealthy industrial, commercial and financial corporations. Of these not much is known of the financial and commercial companies' schemes, which are kept private as a rule; but a good deal is known of the municipal and similar schemes, and of the Railway Funds. The county and municipal authorities have hitherto been left to seek such special powers as they may think fit, and the consequence is an existing lack of uniformity which is quite bewildering, and into which a strong departmental Committee is now enquiring. The great Railway Funds, as we know from the report of Lord Southwark's Committee, present a chequered history of deficiencies for which the Companies have made themselves liable, and of grievances which are believed still to persist.

Of the numerous class of employees in my final category very little is known. They are either unprovided with pensions altogether, or they are dependent on such quasi-eleemosynary recognition as the financial capacity of the employer may permit of, or they are dependent on mutual funds unguaranteed by any external resources. After studying the history of the Railway Funds I doubt if many such unsupported mutual funds can be actually in existence and functioning normally to-day. One cannot, however, prove a negative. Such societies, if existent, are private institutions of which it is impossible to get complete and trustworthy *data*. I have made an effort to hear of such funds, but without success.

It has been obvious for some time past, that over all Mutual Pension Funds, whether supported by external guarantee or not, there has been hanging the shadow of the gradually growing longevity of annuitants. To this cause of anxiety must now be added, as a consequence of the war, the increase in salary scales and the depreciation of securities, of which the former is probably the more important. To investigate the concrete effects of these adverse influences on the finances of mutual funds—particularly those giving benefits based on salary percentages—is beyond the scope of this article, but there is clearly a need for this to be done by someone. In particular, the aggregate deficiency on the Railway Funds will reach very large figures if new valuations be made with a view to nationalization.

It is not a cheerful story. As soon as we get outside the

zone of free State pensions there seems to be running through each of the other broad categories an increasing and now fairly general recognition that contributions from the beneficiaries are necessary ; and it is possible to trace also a general demand for some form of death benefit. All the evidence points to this aspiration having received insufficient recognition from employers in the past. As regards the best methods of attaining the desired ends there seems to be in recent schemes a tendency to resort (perhaps one might say to revert) to what our American friends call "contractual methods", *i.e.*, the money purchase principle ; though this applies more to the category where employers are poor or fleeting. Where guarantees are available and service is more settled, there is evident reluctance to dispense with a scale of pensions bearing relation to salary, and of course for preference final salary.

From this summary of the position to-day, as to which it is difficult to dogmatize but for which a good many facts can be cited in support, I pass to an important matter belonging more to the region of opinion. In all the categories the various services, even the State services with free pensions on the most generous scales, give evidence of discontent. Why should this be so ? My own explanation is that there is a main fundamental question lying at the root of all pension questions to which insufficient attention has hitherto been given. It is involved in the contingency, to which I have already made reference (*see* pp. 116 and 235), of migration from one service to another.

The primitive pension apparently originated in the benevolent wish of an employer to provide for the old age of an employee who had long served him faithfully and could work no longer. But the benevolence of the wish vanished if the employee sought to better himself by transferring his service elsewhere. In that event the faithful servant found he must take his fate in both hands and cast himself on the world or on his new employer; but the old one washed his hands of him and even was glad sometimes to be rid of an accruing liability.

This idea of a pension seems primitive and rudimentary to-day, and more adapted to patriarchal times than to the post-war period. It exists, however, and is still firmly entrenched. I need not pile authority for this statement. The typical pension fund almost invariably provides that on withdrawal the benefit is limited to the member's own contributions.

As evidence for the prevailing conception in services where

there are no contributions, we may turn to the Report of Lord Courtney's Commission, which carefully reconsidered and reviewed the general principles underlying Civil Service Pensions, as well as the question of adjusting their form to meet new needs. Here, we find, Lord Courtney and the majority of his Commission, while they admitted "that a deferred pension is remuneration for services as much as an immediate money payment," proceeded to lay down the general principle that the pension is, in part at least, "remuneration for continuity of service contingently payable on the continuity being maintained during "a defined period, and not accruing from year to year as an "indefeasible interest." The minority, consisting of Sir Ralph Knox and Sir Edward Brabrook, were still more conservative. They adhered to the primitive conception of a pension as the reward of long and faithful service, and objected even to the foregoing limited recognition of the "deferred pay" principle.

Now the Civil servants, on their side, are very conscious as they get older of the grip of the system in which they are held. The recent increase in prices, with no adequate increases of salary in the higher ranks, has emphasized their position of dependence. We have, further, the testimony of our own expert on pension questions, the late Mr. H. W. Manly, that he always recommended employers to provide free pensions, the chief reasons being that superannuation after faithful service should be the aim, that contributions introduced claims for other benefits, and that free pensions would pay for themselves by making the service popular and keeping salaries moderate.* If Mr. Manly could have survived to feel the new impulses now current I doubt if he would retain those views to-day.

The aspiration to-day is not only for free pensions, but for freedom. Old thralls are being thrown off. *A priori* reasoning points rather to a universal principle that every year of service, given by high or low, should carry with it not only the living wage but also its proper quota towards the ultimate old age provision of the worker; and that such quota, once earned, shall enure as a vested right.† This principle already exists in the Federated Universities Superannuation System, and it is adopted (see p. 236) in the new Carnegie scheme for American Colleges.

* See *J.I.A.*, vol. xxxvi, p. 258, and vol. xlviii, p. 34.

† This is what has been known as the "deferred pay" principle when that term is applied in its strict sense; but it has often been used loosely, and I think it better now to use a more exact phrase.

It is found in embryo in the National Health Insurance Act, where every contribution paid by the employer is irrevocable and final. It is, however, independent of the method of providing the contributions. Though we have seen a tendency to divide these between employer and employee, it may be applied equally if they are provided wholly by the employer.

Supposing such a principle established, let us try to visualize some of the sociological results. Many say that employees need not fear in the end the burden of their own contributions, if any. Mr. Carson Roberts has remarked "there is no such thing "as a non-contributory scheme."* When once the pension is secure the free play of the market will determine the salary; and what more can the able official wish? Many employers, and especially the State, would have to pay higher salaries to retain the able men; but why not? There might be other effects. Less able men could be dispensed with more easily if they had pension rights. The net effect to the employer would undoubtedly be a fresher and healthier organization. For the State, above all, any principle that would promote an interchange of officials with the real world outside is to be welcomed. Business and administrative life would alike be quickened and every worker would find increased opportunities either for advancement or for leading the life for which he is suited.

Actuaries in the past have worked well at formulas for measuring the protean benefits and contingencies of existing pension funds and systems—largely self-developed. I hope the Institute may follow the impulse given by our President in his inaugural address, and now seek to be the leader and inspirer of true principles rather than the humble handmaid busy with the evaluation of self-evolved and fortuitous benefits. We can easily see what actuarial results would flow from a general recognition of the broad principle of vested pension rights. While the forms of benefit would remain manifold and the methods of providing contributions as between employer and employee be various, strict provision in advance would be necessary as well as strict provisions for apportioning individual vested interests and for maintaining actuarial solvency. These in turn would probably rule out all salary percentage plans and dictate resort to contractual methods. In other words the money purchase principle would prevail, with definite benefits assured either by ordinary life offices or by a system of "approved societies." We

* *J.I.A.*, vol. xlvi, p 373.

should have to say farewell, if by some a sad farewell, to the absorbing actuarial work based on unknown rates of salary progression, on unknown future benefits, on unknown future contributions, and on unknown rates of withdrawal, and the unknown profit thereon. We shall still be left, however, by way of some consolation, with the future of the rate of interest, the future rate of pensioners' mortality, and the future rates of disablement. There will be plenty here to occupy us for some time to come. As regards disablement in particular there is much work to be done before we can decide whether it is best provided for in combination with superannuation or separately; and if separately by what methods.

It is significant that since the last number of the *Journal* appeared, the opposing principle of free pensions for continuous service has received a fresh lease of life in this country through the passage of the Bill (*see* p. 107) for superannuating the whole of our elementary and secondary school teachers. A measure to raise the status of such an important class, and to improve the education of the country, would be matter of rejoicing if clear evidence had been presented of the soundness of its underlying principles. This Bill, however, was hastily conceived and passed into law without any such evidence being adduced in the expiring days of the late House of Commons, which had long been swimming in a sea of borrowed money, seemingly oblivious of the meaning of figures.

To save space I must refer the reader to the Act itself for the detailed provisions, which are complicated; but the broad effect is to apply the Superannuation Act of 1909 (the salary basis being altered to the average amount received in the last five years of service) to the teachers of all grant-aided institutions below university rank, and to some extent to the teachers of outside schools as well. Speaking generally, all past service is to count for pensions; and existing teachers are to receive in addition the accrued benefit to which they may be entitled in any existing fund. As in the case of the Civil Service, there is no benefit whatever on withdrawal before becoming entitled to a pension. In view of our President's plain speaking on the question of State pensions, it may not be out of place if I record briefly here the facts about the passage of the Bill.

In opening the debate on the second reading, Mr. Fisher apologized for the fact that, though he was speaking on a Monday,

the Bill had not been issued until the preceding Saturday. Most members, therefore, could not have seen it until the day of the debate. After referring to the existing inadequate provision for elementary teachers, the Minister spoke as follows :

"There is no form of pensions whatever for teachers in secondary "schools, although here and there there are local pension schemes. "The absence of such systems led to the appointment in 1912 of a "Departmental Committee to Report upon the best system by which "provision can be made for the superannuation of teachers in "secondary and technical schools and institutions, schools of art, "colleges and schools for the training of teachers, pupil teacher "centres, and other schools and institutions not being universities "or university colleges aided by Grants from the Board of Education. "This Departmental Committee reported in favour of a system of "insurance for full-time teachers in secondary schools, resembling "the system adopted by the Federation of Universities and supported "by contributions from the teachers themselves and from the "employers, to be supplemented by assistance from the State in the "shape of superannuation and disablement allowances. The Report "of this Committee was published, and although widely commented "upon at the time, has never been given effect to. The War "supervened, and the hopes which had been created by the "publication of the Report have not yet been realized. In these "circumstances, and in view of the great development of education "which we expect to ensue from the recently-passed Education Act, "the Government has come to the conclusion that it is essential at "the earliest possible moment to bring under one State pension "scheme all qualified teachers in aided schools of all kinds below "those of university rank.

"Let me say, in the first instance, that the scheme proposed in "this Bill is no niggardly scheme. Its generosity has been widely "and freely acknowledged by all who have studied the White Paper "and who are interested in the welfare of the teaching profession. "I think the House will realize that if the pension scheme for "teachers is to err it should err on the side of generosity. Teaching "is one of those professions which demand the preservation of a "buoyant temper and of a fresh outlook under conditions which too "often make for deadly monotony. Since the State has undertaken "to make provision for the teaching profession it is for the State to "find some means of relieving those school teachers who have "reached that period of life when vitality is lower and when the "spirit tending to usefulness is almost gone. Seeing that the "salaries of teachers generally are on a modest scale, this affords "additional reason why the pension should bear a high ratio to "salary. The scheme, generous in its terms, is non-contributory. "I gathered from some observations from my hon. friend the "Member for the University of Glasgow and Aberdeen University "(Sir H. Craik) while the Financial Resolution was being discussed, "that he was inclined to prefer a scheme under which contributions

"should be demanded from teachers and from their employers. I "do not wish in any way to disparage the value of contributory "schemes. They have their place in the national system. But a "contributory scheme to which the State makes a contribution is "open to certain objections, the force of which is increased in direct "ratio to the size and complexity of the scheme. If you have a "contributory scheme it may be worked by an insurance company, "and if it is worked through an insurance company you are at once "confronted with the objection that public money is going in "dividends to the shareholders of those companies. You are also "confronted with the objection that the Minister of Education will "be besieged by different insurance companies pressing on his notice "the advantages which they are enabled to offer to their clients. "You will be confronted with the objection that the State is "subsidizing and guaranteeing a private company, and these "objections become, of course, seriously aggravated when the sums "involved are large.*

"If you do not go to the insurance company, and if you compel "the teachers and the employers of teachers to make their "contributions to the fund, then you are open to another set of "objections. Teachers know that such a fund earns a comparatively "low rate of interest, and they know that the benefits to be "obtained from such a fund compare unfavourably both as to "amount, elasticity and variety of options with the benefits which "may be obtained from an insurance company. The teacher will "say with some reason that if he is to be asked to contribute his "money—to make a contribution from his modest salary in order to "secure provision for his old age—he should at least be allowed to "take that money to the quarter in which it will earn the best rate "of profit. There is another objection. If you have a fund, then "that fund must be subjected to periodical valuation. I have a "very great respect for the sombre science of the actuary. But the "science of the actuary is not an exact science, and whenever a fund "of this kind comes up for revaluation there will always be disputes "as to the rate of mortality, and there will also be disputes as to "the rate of interest. Again, I say, the larger the operation the "more complex it becomes. We have decided that this scheme of "superannuation shall be non-contributory."

Mr. Fisher proceeded to repudiate the suggestion that the Bill would increase the bureaucratic tyranny of the Board. The suggestions that had been made in that direction were, however, limited to the effect of the Bill on schools at present independent of the Board, and its possible effect in forcing them to become

* The companies might well consider whether and how they should contest this supposed principle, which has been invoked before (Cd. 7365). The Government to which Mr. Fisher belonged must have paid many hundreds, if not thousands, of millions to armament and equipment companies—concerns which certainly do not return the whole or greater part of their profits to their customers as do life offices.

"grant aided," and thus to enter the pension scheme. I cannot find that the fundamental question whether free pensions would or would not in the long run be for the advantage and add to the dignity of the profession as a whole was ever discussed in the way that has been common in America, or even raised by anyone. The claims put forward for the Bill were: (i) that it would promote the unity of the teaching profession; (ii) that it would improve the teaching; (iii) that it would attract the army of men and women teachers who would be required for the purposes of the new Education Act.

Sir James Yoxall, a representative of the teachers, subsequently asserted in the debate that the justification for the Bill was that the Board would not otherwise get the teachers they wanted; and I fancy that is the real reason of the Government for this hasty legislation. This view is confirmed by a statement by Mr. Fisher who, in resisting in the Committee stage of the Bill an amendment for adding to existing pensions, said:

> "The broad principle of the Bill is that it is an attempt to make "the condition of the teaching profession attractive, and to bring "into the profession more teachers in the future. The Bill is "founded on broad considerations of public policy, and it has for its "purpose the securing of a certain definite public advantage. It is "not designed to give benefit to individuals unaccompanied by service "of public advantage. Consequently this Bill is not concerned with "those who have already left the service in Grant-aided schools, and "who will never return to the service, either because they have "migrated to other occupations or because they have retired from "active work."

So far as fundamentals are concerned the only reasons for setting aside the departmental Committee's contributory scheme of 1912 for secondary teachers are contained in the unconvincing statement by Mr. Fisher which I have already quoted; and as regards the private members, twenty in number, who took part in the second reading debate, all with one accord encouraged the expenditure and asked for an extension of it in one direction or another. One or two members had some misgivings on the subject of cost and were referred to the debate on the Financial Resolution.

I have therefore referred to this earlier debate and find that the only information as to the cost of the measure given to the House by Mr. Fisher was as follows:

> "There is one question which has been asked by the Hon. "Member for Somerset (Mr. King), which is germane to the financial

"resolution which we are now discussing. I have been asked to "furnish an estimate of the cost. Of course, hon. Members "will realize that the cost of the Pensions Bill, or any system "of pensions, cannot be very accurately estimated. There are a "number of uncertain factors in any preliminary calculation, but I "will, however, offer the House an estimate which will, I think, be "as good as can be provided at the present juncture.

"There are at present about 100,000 teachers serving the State "under the existing law, and this Bill will increase that number by "about 70,000, and it will also increase the amounts of the pensions "to be granted in the future, besides giving other benefits in addition. "The existing Pensions Scheme now costs us £256,000 yearly, and "would probably cost in ten years' time, if left undisturbed, about "£428,000. The Pensions Scheme now proposed will probably, in "ten years' time, cost about £2,000,000 per annum more than this. "I ought to add that, in giving this estimate, I make no attempt to "forecast the rise in salaries which may take place in ten years' time. "I have endeavoured to take account of the rise which has been "going on through this year but not beyond that. Any further "increase in salaries will ultimately bring with it a corresponding "increase in pensions.

"I have taken account of the increase in the number of "pensionable teachers which will immediately result from the "inclusion within the pensions system of 70,000 teachers who are "now outside the system, but I have not attempted to forecast what "may be the increase in the number of teachers in this country due "to the extension of our educational system. As that grows so "must also the expenditure grow, both on salaries and on pensions. "To carry the forecast beyond ten years would involve too many "problematical results. My hon. friend asked me for an estimate "twenty years hence, but that would be very difficult to give. On "the whole the best indication I can give of the expected cost of "the present measure is to say that at the present moment the "salaries of teachers in grant-aided schools amount to something over "£20,000,000 a year, and on those teachers and those salaries we "expect the pension system ten years from now to cost something "like £2,500,000 per annum, of which £2,000,000 will be new "money, and the remainder is what the present system would cost "if left unaltered. That is as far as I can go at the present moment, "and I hope that indication will satisfy my hon. friend opposite."

Our science may be sombre, and it may be inexact, but the Government Actuary could upon request have furnished the House with something much closer than this. Mr. Fisher states one interesting fact, namely, that on the basis of present establishments the salary list affected will be over £20,000,000 a year. I take this sum though it looks low for 170,000 teachers. On establishments not rapidly growing the "non-effective" vote for pensions on the Civil Service scale has usually varied between

20 and 30 per-cent of current salaries, so that an ultimate charge to the state of £5,000,000 a year will be nearer the mark than the £2,500,000 given to the House as the probable cost in ten years' time. But, as he frankly said, Mr. Fisher made no allowance for the certain growth of the total salary list owing (i) to increases of scale; (ii) to normal increases of establishments; and (iii) to the special increases that will be necessitated by the new Education Act. It is therefore clear that the future expenditure under this legislation must amount to a very large sum.

The Committee stage of the Bill need be referred to only so far as general principles were touched upon. The principal debate centred in a somewhat general desire to extend the Bill to outside secondary schools which, though not grant-aided, were admittedly efficient. The Minister himself proposed an amendment giving the Board power to make rules which would include many of these cases. Sir Henry Craik, the only vocal member who really weighed the fundamental principle of the Bill, again spoke as follows:

"I warn him (Mr. Fisher) that, however small this Bill, and "however little attention is given to its various stages, he is taking "a very new and a very long step forward towards making teachers "in public schools into Civil servants, and I object to that on the part "of the teaching profession. I think that sooner or later this will "work evil in the teaching profession. Why are you to take one class "of the community more than another, unless they are to be considered "as Civil servants in the service of the State, and give them pensions "without any contribution? There is no other class who hitherto "have been pensioned on a non-contributory system. This matter "does not deal only with England. It affects also Scottish teachers "very largely. Hitherto it has been the case that Scottish and "English teachers were on an equal footing. . . . . Now you are "digging a gulf between the two sets of teachers. A teacher cannot "change from England to Scotland without dislocating entirely the "scheme of pensions for which he stands. In Scotland we have, so "far, only contributory pensions. I am quite aware that teachers "are not a very reticent class in Scotland if they have any grievance, "and they have not, so far as I am aware, asked that the contributory "system should be entirely abolished. I have always insisted on "liberal aid being given, but that it should be started on a principle of "small contributions, and I do not see why you have suddenly started "an entirely new scheme for England, which is to leave Scotland out "altogether, and which is to exclude from the category for which "provision is made one class of schools which do not come within "the hard-and-fast official regulations. That is the centre of my "doubts about your Bill, and it is not removed by the Amendment

"of the Right Hon. Gentleman, though it is the only Amendment "which has been introduced to alleviate this hardship. I do not "think that it is sufficient to do away with very many of the "objections which can be urged against this Bill."

Opposed as he was to the principle of the Bill, Sir Henry Craik, as I think a little unfortunately, said that if passed at all it should be extended to all schools. The battle raged, in fact, round the fear of the bureaucratic tyranny of the Board. He was followed by Sir William J. Collins, who was a member of the recent Committee at King Edward's Fund on Hospital Pensions, and who spoke as follows, after welcoming the Bill:

"As has been pointed out, in the matter of principle this Bill "has made a most important change in the whole method of retire- "ment legislation, and I think that that might be given as a reason "for not pressing the Right Hon. Gentleman to go much further "than he has done in the case of these non-Grant-aided schools. "After all in 1912 a committee was appointed to consider the "pensions of secondary school teachers, and the Right Hon. Gentleman "has told us that he threw over the proposal of a federated "university scheme on the contributory principle, and, under this "Bill, pensions are being given out of moneys provided by Parliament "to persons who are not in the employment of the State without "any contribution either from the beneficiaries, or from those who "employ them in the shape of local authorities or management. "These, indeed, may be described as revolutionary principles, and "very generous principles, in the matter of pensions, but I think one "can hardly press the Right Hon. Gentleman to go much further "when we are dealing with these private schools which are still less, "as it were, connected with any State organization. If the teachers "in them are to enjoy pension rates on the Civil Service scale without "any contribution whatever, if that principle be conceded, I can "hardly see where the development on these lines will stop. You "must remember that the schemes of the police, the Poor Law, the "Asylums officers, the Metropolitan borough councils are contributory "schemes, and obviously in the case of the municipal employés and "the borough and county councils, about whom a committee is "sitting, the principle in this Bill will be cited as a model for future "pension schemes. I confess that to press the Right Hon. Gentleman "further in the matter of these private schools than he has already "gone or to widen the limitations he has prescribed in this "Amendment is scarcely fair. . . . I do not think in the Committee "stage or now we have heard any estimate of the final cost out of "the public revenue of pensioning these teachers in private schools. "For these reasons I do not think the Right Hon. Gentleman should "be pressed to go further. He has made generous provision for "the private teacher and I hope this Bill will satisfactorily pass "into law."

Mr. Dillon not only supported the Bill, but pressed for its universal application as opposed to leaving any discretionary power with the Board. As an old parliamentarian he may well be listened to on this point :

"All experience in this matter goes to show that when you "introduce measures of this character which set up a privileged "class the results of the operation of these measures go far in excess "of what their originators dreamed of. They set up a tendency "which grows and becomes irresistible, and although it is only "recently that the Minister of Education has got control of a "Government Department he must know the inevitable law, like "one of the great laws of science, that all the Government "Departments are consumed with a desire to eat up everything that "comes their way. I am not blaming them. It is of their essence "and nature and Government Departments universally—I never "knew an exception—imagine that their methods are the best, and "their whole desire and tendency is to gather everything under the "machine, run it all in accordance with the machine and get it "under their control."

After similar appeals for extension, Mr. Fisher brought the discussion back to a sense of proportion by remarking that, while the original Bill would apply to 170,000 elementary and secondary teachers, there were left outside only 2,600 teachers in 105 schools who were on the Board's efficient list, and that his object was to try and provide for some of those. If the whole were included the annual cost in 10 years' time would be about £90,000. His proposal was then agreed to. I think nothing else of main interest occurred in Committee apart from the general endeavour to secure extensions—especially in the case of administrative officers of education authorities ; and I would have made my quotations still shorter, but for the side-lights thrown on important questions of principle.

What is the bearing of the new Act on the general principle of vested rights, with which in this note I am more particularly concerned? I cannot see that it carries any weight at all in theory, whatever it may do in practice. If it is to the temporary and pecuniary advantage of the State to bribe or induce entrants to a profession by the promise of free pensions, and thus secure them in a grip for life, then surely it must be to the advantage of the profession to cry " *timeo Danaos* " and refuse the proffered gift. As I have said, the only vocal member of Parliament who faced fundamental principles was Sir Henry Craik, and he on the second reading remarked : " I should be false to the whole

"work of my life if I did anything which would delay and "hinder a real benefit to the teacher. As I have urged over and "over again, be liberal in your salaries and pensions; but I am "not quite sure that this benefit, accompanied as it is by a "transformation of the whole profession into what will come "very nearly to a Civil Service system, will be in the long run "for the real advantage of that profession or for the advantage "of education."

To the onlooker it would seem that prior to this legislation the teaching profession was in a strong position. The State was faced with the apparently impossible task of securing the teachers it required on the basis of existing inducements. By combination the teachers could have raised the whole level of remuneration of their profession; while the superannuation system recommended by the Departmental Committee of 1912, or something similar, would have secured to them the vested pension rights, and thus at the same time their liberty. As Mr. Fisher happily remarked "the casuistry of life is infinite", and one must not dogmatize without full knowledge; but as the case stands we must refuse to accept this new Act, based as it is on opportunist reasoning, as settling the question of principle whether pension rights should or should not vest from year to year. Meanwhile it seems to me that the University staffs in this country, who do not come within the scope of the new Act, but who have already succeeded in establishing the principle of vested rights, would be wise to regard the preservation of that principle as fundamental in any steps they may take to improve their own position.

Whatever may be our opinions on the merits of this question of principle it is clearly matter for regret that Parliament should have failed signally to realize the importance of discussing it and dealing with it advisedly. The need for some clear thinking on it still exists, and it is thus that the points involved present themselves in summary to my mind:

(i) If men and women are to do their best work must they not be free?

(ii) If they are to be free must not pension rights enure and vest from year to year?

(iii) If the answer to these questions is in the affirmative economic enquiry should follow into a remaining question, namely: On whom, whether employers or employees or both, does the cost of pensions really fall?

In other words, we need to determine clearly whether, as is commonly supposed, pensions can only be paid for in the long run by the labour of the employees. If that be so and if the cost really falls on the latter, we shall then be on firm ground in asserting that they cannot sacrifice freedom and accept free pensions, where employers hold the power of the purse, without economic detriment.

The American-Canadian Mortality Investigation, 1900–1915.*

[Communicated by Mr. Arthur Hunter, A.I.A., F.F.A., Chairman of the Committee on Mortality Investigation.]

Synopsis of Vol. I of the Report.

THERE are two tables in use in the United States for the calculation of premiums and of reserves, namely, the "Actuaries' Combined or Seventeen Offices", and the "American." The former, published in 1843, has been little used in recent years except by companies which assumed 4 per-cent interest. The principal reason for the "Actuaries'" Table remaining in use so long is that it is the basis for valuation in States in which 4 per-cent interest is permitted. The "American," with interest at 3 per-cent or $3\frac{1}{2}$ per-cent, is the reserve standard in most of the States and is the basis of premiums in the principal American companies. The table was completed in 1860 and was based largely on the experience of the Mutual Life Insurance Company. The rates of mortality are considered as "ultimate," but the American Table at the younger ages is known to be distinctly higher than the ultimate experience. As the reserves based on that table are considered sufficient and as the premiums based thereon result in a satisfactory margin of safety, there has not been a disposition to make the radical changes which a new table might necessitate. These might involve not only a change in the premium rates and in the reserves, but also in the provisions of the laws of the State of New York with regard to initial commissions to agents, the measurement of the company's first year's and total expense allowances and the rule for determining the maximum new business permitted.

The National Convention of Insurance Commissioners in

* Published by the Actuarial Society of America.

August 1911 requested the Actuarial Society to "turn its "attention to the construction of new mortality tables covering "the general experience of the companies among normal lives." It was then pointed out by the Society that the new table would probably show slightly higher reserves and that the premiums charged by the insurance companies would not necessarily be affected by such a new table because a larger loading might be required. A larger percentage of loading would doubtless be necessary in the case of the companies issuing non-participating policies with a loading of from 0 to 5 per-cent on the net American $3\frac{1}{2}$ per-cent premiums. These companies expect to pay part of their expenses out of the savings in mortality which would largely be eliminated under the new table.

The Actuarial Society expressed its intention of preparing a new mortality table after the Medico-Actuarial Mortality Investigation had been completed. In 1914, before the publication of the last volume of the report on the M.A.M.I., the Insurance Commissioners again asked the Society "to con-"struct a mortality table which is in keeping with the ascer-"tained experience of American companies." In 1915 the first meeting to make the plans for the investigation took place. The Insurance Commissioners appointed a committee of Insurance Department Actuaries to co-operate with the committee of the Actuarial Society, while the latter invited the American Institute of Actuaries to appoint a committee of co-operation. The work has been conducted by the Actuarial Society and the other two bodies have given their advice and assistance. While the main purpose of the investigation was to obtain "select" and "ultimate" tables based on the experience of men resident in the United States (American men), advantage was taken of the opportunity to obtain corresponding tables for men resident in Canada (Canadian men), and to investigate the mortality among women. These results are given in Volume I, while in Volume II will appear the report on other investigations such as the mortality by plan of insurance among American men, by groups of States in the United States and by provinces in Canada.

As an investigation based on recent data was desired, it was decided to take the experience during the years 1900 to 1915 inclusive. This included the issues of 1900 to 1914 carried to the anniversary of the policies in 1915, and also the issues prior to 1900, if continued in force until after the anniversary of

the policies in 1900. Thus, a policy issued in 1895 was observed from its anniversary in 1900 to the anniversary in 1915, or its previous termination.

The investigation is based on amounts insured, but policies issued at the same age on any individual life for more than $100,000 were treated as $100,000.

If all the data of the large companies had been taken, they would have had a preponderating influence in the investigation, and accordingly a rule was adopted whereby the larger the company, the smaller would be the percentage of its data to be contributed.

For the benefit of students the report gives the reasons of the committee for adopting the several rules such as those with regard to policies (*a*) dated back, (*b*) changed in amount and in plan, (*c*) lapsed and reinstated. The reasons are also given for omitting policies re-insured in bulk, or issued on the group insurance plan and those issued incontestable from date.

The mean duration method of determining the duration of the insurance was used.

Probably fewer refinements in technical treatment were made than in any similar investigation, but it is believed that the methods used give as accurate results as are necessary for the purposes of the tables.

A study of the methods used in collecting and in recording the data shows that the companies in general gave their share of the data on "policy sheets", and that "perforated" cards were prepared by the Bureau established by the Actuarial Society to carry on the work. Both the perforating and the sorting machines are developments of those used in the M.A.M.I. The tabulating machine, in addition to making summations of different columns, printed a record of each card, giving such items as year of issue, age at entry, duration, sex, mode of termination, plan of insurance, habitat at date of application for insurance, and amount of insurance.

Extent of the Statistics.

Fifty-nine American and Canadian companies, representing 95 per-cent of the regular insurance in the United States and Canada, contributed to the investigation.

From the following table, the extent of the data may be judged:

	American Men	American Women
Total exposures ...	$26,003,321,900	$787,819,500
Dead	436,345,200	10,878,500
Average exposure...	6·88 years	6·18 years
	Canadian Men	Canadian Women
Total exposures ...	$2,743,665,100	$69,144,100
Dead	29,400,800	672,500
Average exposure...	6·05 years	6·37 years

AMERICAN MEN.

A number of tests showed that the effect of medical selection apparently did not extend beyond the fifth policy year. At the younger ages the mortality was much lower than in the American table, the ratio to the American table, for example, at attained age 30 was 51 per-cent. It was accordingly to be expected that the mortality during the early policy years would not show as great an effect of selection as was generally assumed.

In the following table the ultimate q of the new experience of American Men—to be known as the A.M.—is compared for decennial ages with the q of four standard tables :

Comparison of Rates of Mortality with A.M.$^{(5)}$

Attained Age	Ratio to $O^{M(5)}$	Ratio to American	Ratio to M.A. ultimate	Ratio to U.S. Males (1910)
25	63%	53%	90%	77%
35	57	55	96	58
45	66	69	103	61
55	84	95	112	82
65	96	103	106	95
75	99	96	98	97

The foregoing percentages show (1) that there has been a marked improvement in the mortality among insured lives at the younger ages, but little, if any, at the older ages ; (2) that there is not a wide difference on the whole between the Medico-Actuarial Mortality Table (ultimate) and the American Men ultimate table ; (3) that there is a much lower death rate (ultimate) in the insurance companies than in the population of the United States except at the older ages when there is little difference.

A feature of the ultimate curve of mortality is that neither the Gompertz nor Makeham method is suitable prior to age 50, and it is necessary, on account of the flatness of the curve at the younger ages, to apply a supplementary curve. In the report are given at quinquennial ages the results of graduations by Spencer's 21 Term Formula, King's Osculatory Interpolation, Larus' 23 Term Formula, G. F. Hardy's Friendly Society Formula, a Makeham graduation with a supplementary curve, and a Gompertz graduation with a supplementary curve. The method finally adopted was based upon Makeham's second modification of Gompertz's Law with a supplementary curve.

The same method was applied to the graduation of the first five insurance years with modified values of the constants other than c for each duration. A short table is now given of the graduated rates of mortality at decennial ages at entry :

American Men.

Age at Entry	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	$q_{[x]+3}$	$q_{[x]+4}$	$q_{[x]+5}$
15	·00247	·00324	·00341	·00355	·00372	·00392
25	·00293	·00384	·00402	·00412	·00427	·00446
35	·00316	·00429	·00457	·00480	·00523	·00584
45	·00482	·00690	·00782	·00866	·00992	·01158
55	·00994	·01447	·01712	·01951	·02271	·02668
65	·02252	·03267	·03931	·04517	·05263	·06147

In the M.A. Mortality Table, based on policies, the effect of selection was not apparent for more than five years, and the same is true of the new table. In the latter the ratio of actual to expected deaths by the ultimate table for ages at entry 15 to 34, is 100·1 per-cent for the sixth to the tenth insurance years and 98·7 per-cent for the eleventh to the fifteenth insurance years. There is no evidence that the ultimate rates should have been prepared by including more policy years at the young ages or fewer years at the older ages at entry.

As the last death in the American Experience Table is assumed to be at age 96, the custom of the companies is to pay the face amount of the insurance at that age. The history of these cases was traced by the companies to the date of death, or the policy anniversary in 1915.

The net premiums under the American Men Mortality Table (ultimate) are naturally lower than under the American Experience Table. At age 20 this amounts to 18 per-cent, and

at 60 to 1 per-cent on the Ordinary Life plan. Comparing the new table with the $O^{M(5)}$, the corresponding percentages are 17 per-cent and 2 per-cent. On the 20-year Endowment Insurance plan at age 20 the $A.M.^{(5)}$ is $5\frac{1}{2}$ per-cent lower than the American and 4 per-cent lower than the $O^{M(5)}$ while at age 60 there is practically no difference in the premiums under the American and $A.M.^{(5)}$, but the $O^{M(5)}$ is 2 per-cent higher than the $A.M.^{(5)}$

The difference between the mean reserves on the $A.M.^{(5)}$ and the American varies with age at entry, duration of policy and plan of insurance, although they are generally higher under the former than under the latter on the Ordinary Life plan. A valuation of the entire business of an old established company showed 2 per-cent higher reserve on the $A.M.^{(5)}$ than on the American basis.

It is rather rash to express an opinion with regard to the chance of the new table taking the place of the American for the calculation of premiums and reserves. The chance is less than it was before the influenza epidemic. The deaths during the last three months of 1918 were in most companies approximately the same in amount as during the first nine months of the year—in other words the death rate for October, November and December 1918, was three times the normal. In these months the deaths in the population from influenza and epidemic pneumonia was about 4·5 per 1,000 inhabitants.

Canadian Men.

As may be seen from the following comparison of the rates of mortality for the sixth and succeeding insurance years combined the mortality among Canadian men is much lower than that shown by the $O^{M(5)}$, the standard for valuation in Canada :

Comparison of Canadian Mortality Table $C.M.^{(5)}$

Attained Age	Ratio to $O^{M(5)}$	Ratio to $A.M.^{(5)}$
25	64%	102%
35	53	93
45	58	88
55	71	84
65	83	87
75	93	94

Except at the young ages the ultimate mortality among Canadian men is lower than among American men and the same result was found to apply generally to the first five policy years.

The method of graduation was the same as that applied to the American Men Table, but there were greater percentages of deviation in the expected deaths as the material for Canadian men was only 10 per-cent of that for American men. In the following appears the graduated rates of mortality for decennial ages at entry :

Canadian Men.

Age at Entry	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	$q_{[x]+3}$	$q_{[x]+4}$	$q_{[x]+5}$
15	·00195	·00307	·00342	·00370	·00393	·00412
25	·00254	·00365	·00394	·00411	·00423	·00428
35	·00280	·00380	·00414	·00453	·00495	·00533
45	·00469	·00582	·00659	·00761	·00877	·00987
55	·01011	·01186	·01386	·01657	·01964	·02258
65	·02425	·02783	·03303	·03996	·04773	·05507

The net premiums on the C.M.$^{(5)}$ are lower than on the O$^{M(5)}$. The mean reserves are sometimes higher and sometimes lower on the former than on the latter, depending on age at entry, duration of policy and plan of insurance.

Mortality among Women.

It was shown in the Medico-Actuarial Mortality Investigation that the mortality among women depended upon their conjugal condition at the date of application for insurance. The lowest ratio of actual to expected deaths was 81 per-cent of the Medico-Actuarial Table (M.A.) among spinsters and the highest 126 per-cent among married women whose husbands were the beneficiaries. When the data were divided by plan of insurance the relative mortality among the four classes of women differed only slightly from the experience for all plans combined. The present investigation does not make a division according to either conjugal condition or plan of insurance, and its chief value therefore is in enabling us to compare the experience by amounts insured among women with that among men.

The ratio of actual to expected deaths by the A.M. Table is 110·8 per-cent for American women for the first to the fifth insurance years, 91·5 per-cent for the sixth and succeeding years, and 94·4 per-cent for the entire experience. The detailed

tables show that the mortality among American women was not as good as among American men in the early insurance years, but was distinctly better in the sixth and succeeding years at attained ages over 40. On the whole the companies have experienced a better mortality on women than on men.

Measuring the mortality on Canadian women by the select and ultimate table for Canadian men (C.M.) the ratio of actual to expected deaths is 117·9 per-cent for the first five insurance years, 93·3 per-cent for the sixth and succeeding years and 97·9 per-cent for the entire experience. The relative mortality was slightly lower than among American women, but this may be due to a different distribution by conjugal condition, or to a different distribution of the insurance between very healthful and less healthful habitat, or to accidental fluctuations. (The amount of material on Canadian women is small, the actual deaths amounting to $672,500).

The statistics show that the experience on Canadian women is slightly more favourable than among Canadian men, but less favourable in the earlier insurance years and at the younger attained ages.

There was no difficulty in choosing a name for the experience on Canadians. As there has not been any table prepared in Canada, based on either the population or insured lives, it was obviously proper to use the word "Canadian" as a designation. When it came to selecting a name for the American experience considerable difficulty was met. The name "American" had been pre-empted by the table constructed in 1860 by Sheppard Homans and D. Parks Fackler, although it did not merit such a comprehensive title. The phrase "United States Life Tables" could not be used because the Government had selected that designation for the experience based on the population of 1910 and the deaths for the three years 1909, 1910 and 1911. There is an Actuaries' Table and an American Offices Table. While not very satisfactory on account of the limitations thus imposed the title chosen was American Men Mortality Table. It is therefore written A.M. and not A^{M}.

The instructions and rules were sent to the companies in June 1916, and the data recorded on policy sheets were received by the Bureau at the end of that year or the beginning of 1917. With a force of less than twenty women the work of perforating over 2,000,000 cards, of sorting and tabulating the data was completed with the help of modern electrical machinery in less

than a year. The calculation of the expected deaths was done by a force of clerks of one of the insurance companies, working after their regular office hours. If the war had not intervened, making it difficult to obtain the services of trained clerks, delaying the printing for many months on account of lack of proper paper and expert type-setters, the volume containing the investigations here described would have been available in less than two years after the companies were asked to collect their statistics.

Two recent Legal Decisions affecting Titles to Reversions. *By* A. H. WITHERS, *Barrister-at-Law.*

The decision in Hill v. Peters [1918] 2 *Ch.* 273.

IT has long been settled law that if A, being entitled to a life or reversionary interest in personal estate held by a trustee, assigns it first to X and secondly to Y, then X and Y are entitled to priority as between themselves in the order of time in which the trustee receives notice of their respective assignments. This is known as the rule of Dearle *v.* Hall. The exceptions to this rule prevent Y obtaining priority over X (1) where Y has not given value for his assignment or is a trustee in bankruptcy of A, &c., and (2) where Y at the time of taking his assignment has notice (actual or constructive) of the assignment to X.

The recent decision of Mr. Justice Eve in *Hill* v. *Peters* says there is yet another exception, namely, where the assignment to X takes the form of a declaration of trust by A in favour of X.

In *Hill* v. *Peters* the material facts were shortly as follows: In September 1897 a reversion was mortgaged to two solicitors, who promptly gave notice to the trustees of their mortgage, and in the following month declared by deed (of which notice was not given to the trustees) that they held the mortgage in trust for the defendant Peters and another. In 1907 the two solicitors, in fraud of their beneficiaries and without disclosing the trust, purported to join in mortgaging the reversion to Mrs. Gwynn (whose executor was the plaintiff Hill) and notice of her mortgage was given to the trustees in 1917. It was held by the learned judge that (1) the rule of Dearle *v.* Hall did not require notice to be given of the declaration of trust, and (2) the assignment to Mrs. Gwynn being in fraud of the beneficiaries passed nothing, and (3) therefore the beneficiaries under

x 2

the prior declaration of trust (of which notice was *not* given to the trustees) had priority over the mortgage to Mrs. Gwynn, of which notice was given to the trustees.

It is impossible to tell from the report in the Law Reports what exactly was the transaction of 1907. Presumably the solicitors did not in 1907 receive payment of their mortgage debt; because if they had been paid off, the rights of their beneficiaries would probably have been destroyed by Section 20 of the Trustee Act, 1893, which provides that "the receipt in "writing of any trustee for any money securities or other personal "property or effects payable, transferable, or deliverable to him "under any trust or power shall be a sufficient discharge for "the same, and shall effectually exonerate the person paying, "transferring, or delivering the same from seeing to the applica-"tion or being answerable for any loss or misapplication thereof."

The learned judge decided definitely that the rule in Dearle *v.* Hall does not apply to a declaration of trust so as to require the beneficiary under the declaration to give notice of his interest to the trustee holding the fund. It is certainly curious that the point seems never to have been considered previously, and there is a good deal of force in the observation that where an equitable interest is lawfully left in the name of a trustee it is unnecessary, on the point of priority, to give notice to the holder of the fund that other persons claim *through* the trustee: such a notice merely supports the right of the trustee to receive the equitable interest left in his name. But on the other hand Courts of Equity usually regard the substance rather than the form of a transaction, and it seems absurd, and opening the door wide for fraud, to say that the rule of Dearle *v.* Hall applies where the document takes the form of an assignment and not where it takes the form of a declaration of trust by a vendor or borrower in favour of a purchaser or mortgagee.

The arguments and judgment in the case suggest that the point would have arisen just the same had there been no formal declaration of trust, *e.g.*, if the solicitors had been trustees of a fund and rightly or wrongly invested it upon the mortgage of 1897 and had executed no declaration of trust. It is difficult to see how Dearle *v.* Hall could apply to such case. Of what assignment could notice be given to the trustees?

The rule in Dearle *v.* Hall having been brushed aside, the way was clear for the application of an exceedingly inconvenient rule, which undoubtedly applies to land and shares, namely,

that if a purchaser or mortgagee of land or shares in a company obtains only an *equitable* estate or interest in the land or shares, then he takes subject to all prior equitable estates or interests in the land or shares, *e.g.*, if the vendor or mortgagor has declared himself a trustee for others and subsequently *makes the sale or mortgage in fraud of his beneficiaries* then, in the absence of special circumstances, the beneficiaries are not prejudiced by the sale or mortgage. The second part of the decision in *Hill* v. *Peters* is that this rule applies to choses in action, as well as to land and shares, in cases where the rule in Dearle *v.* Hall does not apply.

So far as the present writer is aware, this is quite a new point and has not previously been put forward. It has always been considered that one is safe in dealing with the ostensible owner or mortgagee, regardless of any secret trust that such owner or mortgagee may have created or been subject to. It is now decided that this is quite wrong.

The decision is one of the utmost importance to purchasers and mortgagees of reversions; more important indeed than any case that has been decided for a good many years past.

The practical result of the decision, if correct, is that if an ostensible owner or mortgagee (past or present), is, unknown to the purchaser or mortgagee, a trustee for others and acts in breach of trust, then the purchaser or mortgagee takes subject to the rights of the undisclosed beneficiaries. It is absolutely impossible to prove the negative, or to obtain satisfactory evidence on the point or gain protection by enquiry of or notice to the trustees of the fund.

If *Hill* v. *Peters* is correct, its inconveniences may be reduced in some cases by the section above quoted from the Trustee Act, 1893, *e.g.*, if in *Hill* v. *Peters* the two solicitors had been paid off and had given a receipt for their mortgage, their beneficiaries would have lost their security. That section, however, is of somewhat limited extent, *e.g.*, it does not seem to apply where the ostensible owner or mortgagee is only one of several trustees, or where a trustee is selling without any trust or power of sale, or where a trustee gives a receipt or release without receiving payment in full.

The decision In re Pain, 1919, 1 *Ch.* 38.

In this case Mr. Justice Younger decided that a purchaser or mortgagee of a reversion is usually not affected by the reversioner,

after the trustee has received notice of the assignment, commencing and prosecuting litigation against the trustee and being ordered to pay the costs of the proceedings. But if the assignee stands by and allows costs to be incurred by the reversioner *which the assignee could have prevented*, then the costs payable by the reversioner to the trustee are charged on the reversion in priority to the claims of the assignee. Compared with *Hill* v. *Peters*, this decision is of trifling importance; but it is a decision that has to be borne in mind. When a purchaser or mortgagee is made a party to an action affecting the trust, he has to consider whether he can and ought to stop the action. If he cannot stop the action, then he cannot be blamed or made in effect to bear the costs; but if he can but does not prevent the costs being incurred, he allows the proceedings to continue at some risk to himself. This point will trouble mortgagees more often than purchasers.

LEGAL NOTES.

By William Charles Sharman, F.I.A., *Barrister-at-Law.*

Policy issued to cover funeral expenses. Conflict of Claim.

A FURTHER case on the construction to be placed upon Section 36 (2) of the Assurance Companies Act, 1909, is that of *Hatley* v. *Liverpool Victoria Friendly Society*, L.T.R. 118, 687.

This case, which was an appeal from a decision of the City of London Court, dealt with the question as to who was entitled to the proceeds of an industrial policy of life assurance effected by a father on the life of his son. The decision of the Court was in accord with the decision of the Court of Appeal in the case of *Da Costa* v. *Prudential Assurance Company* (*J.I.A.*, vol. li, p. 51).

The facts are as follows:

In 1891, George Lake effected a policy of insurance in the Liverpool Victoria Friendly Society on the life of his son. William Oliver Lake, aged two years. He retained the policy and continued to pay the premiums until his death in March 1915. There was evidence that he gave the policy to his daughter, Mrs. Jenner, and that she paid the premiums after his death. William Oliver Lake died in June 1916, and in June 1917, letters of administration to his estate were taken out by the plaintiff who claimed the policy moneys.

The trustees of the defendant Society made an award in favour of Mrs. Jenner, whereupon the plaintiff sued them in the County Court. The action was dismissed and upon appeal by the plaintiff the Court upheld the decision of the County Court Judge.

In the course of his judgment, Avory, J., said: "All the "authorities which have been cited to us I think result in this, "that the question is whether there is admissible evidence to "show that the father at the time he made the purchase of the "policy in respect of his child intended that purchase to be for "his own benefit or for the benefit of the child. If there is "admissible evidence to show that he intended it to be for his "own benefit, then the presumption that it was for the child's "advancement is rebutted. Now, in the present case I think "there was ample evidence to rebut the *primâ facie* presumption "that this policy was intended for the benefit of the child. "Looking at the terms of the policy itself and at all the "circumstances of the case, it is clear, I think, that the father "took it out for his own benefit, in order to cover the possible "expenses of the child's funeral. That view of this case is "confirmed by a reference to the Assurance Companies Act, "1909, s. 36, the second paragraph of which, in my opinion. "applies to this policy: 'No policy effected before the passing "'of this Act with a collecting society or industrial assurance "'company shall be deemed to be void by reason only that the "'person effecting the policy had not, at the time the policy "'was effected, an insurable interest in the life of the person "'assured.' It is to be observed that prior to this Act there "had been a question, at all events, raised as to whether the "fact of a father being liable for the funeral expenses of his "child gave him an insurable interest in the life of the child. "Then the section a little lower down says: 'If the policy was "'effected by or on account of a person who had at the time a "'*bonâ fide* expectation that he would incur expenses in "'connection with the death or funeral of the assured, and if "'the sum assured is not unreasonable for the purpose of "'covering those expenses. . . . . . such policy shall enure for "'the benefit of the person for whose benefit it was effected or "'his assigns.' I think that applies to the present case, and "that therefore shows, in conjunction with the rest of the "evidence, that this policy, which was one taken out for the "benefit of the father, did enure for his benefit, and therefore

"formed part of his estate. Under those circumstances the "plaintiff had no claim or title to it. On that short ground I "think the judgment of the learned County Court Judge must "be upheld and the appeal dismissed."

Shearman, J., concurred.

Company subject to British and Colonial Income Tax. What is proper deduction from dividend on Preference Shares?

A case of some interest to life assurance officials who have to deal with questions of income tax is that of *Rover* v. *South African Breweries, Limited*, T.L.R. 34, 478. This was a special case stated by consent to decide whether a company which has paid British income tax and also Colonial income tax, and has obtained from the British authorities repayment of the Colonial income tax, is entitled to deduct from a shareholder's dividend the total amount paid on account of British income tax or such amount less the amount of Colonial tax recovered.

It was decided by Astbury, J., that the company could only deduct from the dividend due to the preference shareholders the amount of British tax less the Colonial tax recovered.

The facts are as follows:

The plaintiff was the holder of 500 5 per-cent cumulative preference shares and 1,300 ordinary shares in the South African Breweries, Limited. The directors of the company paid 5 per-cent on the preference shares and a dividend on the ordinary shares.

The company paid income tax on its profits in South Africa at a rate of not less than 1*s.* 6*d.* in the £, and was assessed to British income tax at the rate of 5*s.* in the £ on its assessable profits. Relief was granted under Section 43 of the Finance Act, 1916, to the extent of 1*s.* 6*d.* in the £, subject to proof of payment of the tax in South Africa.

The company deducted income tax at the rate of 5*s.* in the £ from its preference share dividends and 3*s.* 6*d.* in the £ on the ordinary share dividends.

In the course of his judgment Astbury, J., said:

"The Colonial Tax is, *qua* the shareholder in this country, "not a duty charged within the meaning of Section 54, but is an "outgoing of the company, as any other administrative expense, "in the Colony, which has to be satisfied before any profits can "be ascertained and distributed in this country by way of "dividend. The real question is whether the company was "charged and paid 5*s.* or 3*s.* 6*d.* in the £ for British income tax "for the year 1917, within the meaning of Section 54. If the

"former it was right in making the 5*s*. deduction from the "plaintiff's dividend; if the latter it still owes to him the unpaid "portion of the dividend that he became in law entitled to when "the declaration of the dividend was made. The company "contends that under Section 43 the person who pays both "Colonial and British tax is the only person entitled to recover "the 1*s*. 6*d*. in the £, and to keep it It is said that the share-"holder has not paid the Colonial tax, and therefore does not "bring himself within the benefit of the section. The company "also says that if the plaintiff succeeds he and those in his "position will become a privileged class of British taxpayer, "escaping with the payment of only 3*s*. 6*d*. in the £. It is also "urged that as between the company and the plaintiff the "former did in effect pay 5*s*. in the £ for British tax, and that "it was only on proof of this that the right to repayment arose "under Section 43, although the matter was, as between the "company and the Crown, dealt with in account as I have stated. "I think that the company, having paid both the taxes in the "first instance, is the person entitled to the repayment of the "1*s*. 6*d*.; but the question remains whether when it has obtained "it, it can reasonably and in truth be said to have paid or to "have become liable to pay 5*s*. in the £ for British tax. The "whole object of Section 43 was that the payment of income "tax was to be reduced as therein provided, and it is difficult to "see how a company which has paid 5*s*. on the terms that it "should immediately be repaid 1*s*. 6*d*. cash out of it can be said, "after such repayment, to have expended more than 3*s*. 6*d*. If "this is so it follows that the only deduction authorized by "Section 54 of the Income Tax Act, 1842, is 3*s*. 6*d*., and that the "balance, 1*s*. 6*d*., of the dividend is still unpaid or payable."

Appointment of Royal Commission on Income Tax.

In view of the complexity of the income tax laws, a complexity which does not seem to have been decreased by the passing of the codifying Act in 1918, it is interesting to record that the Government have appointed a Royal Commission to enquire into the whole subject. Actuaries will have been gratified to learn that the President of the Institute, Mr. Geoffrey Marks, is to serve on this Commission.

On Certain Inequalities and Methods of Approximation.
By J. F. STEFFENSEN, D.Phil.

IN a previous paper* I have proved an inequality of very general nature, admitting applications to certain actuarial problems. The subject has been taken up by Mr. B. Meidell,† whose starting-point was the well-known inequalities of Dr. J. L. W. V. Jensen.‡ In the present paper I propose to give an account of these investigations, commencing with my own (with a few unessential simplifications), and generalizing to a certain extent Mr. Meidell's and Dr. Jensen's.

Let us assume that two functions $f(t)$ and $\phi(t)$ are integrable, that is, admit an integral from a to b; that $f(t)$ never increases, and that $0 \leqslant \phi(t) \leqslant 1$. Then, putting for abbreviation

$$\lambda = \int_a^b \phi(t)\,dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we can prove that

$$\int_{b-\lambda}^{b} f(t)\,dt \leqslant \int_a^b f(t)\phi(t)\,dt \leqslant \int_a^{a+\lambda} f(t)\,dt \quad . \quad . \quad . \quad . \quad (2)$$

If $\phi(t)=1$ or $\phi(t)=0$ or $f(t)=\text{const.}$ for all t, the two limits in (2) coincide.

Before proceeding to the rigorous proof we may observe that the inequalities (2) are almost obvious. For instance, dividing them by λ, the expression

$$\frac{\int_a^b f(t)\phi(t)\,dt}{\int_a^b \phi(t)\,dt}$$

may be regarded as a *weighted mean* of $f(t)$, the weights being $\phi(t)dt$, and the total of weights λ. But in the mean

$$\frac{1}{\lambda}\int_a^{a+\lambda} f(t)\,dt$$

* "On certain inequalities between mean values and their application to actuarial problems." Skandinavisk Aktuarietidskrift, 1918, pp. 82–97.

† "Note sur quelques inégalités et formules d'approximation." Skandinavisk Aktuarietidskrift, 1918, p. 180.

‡ "Sur les fonctions convexes et les inégalités entre les valeurs moyennes." Acta Mathematica, vol. xxx, p. 175 (1905–6). *See* also an earlier paper (quoted by Dr. Jensen) by O. Hölder in Göttingen Nachr., 1889, p. 38; a Paper by L. Galvani in Rendiconti del Circolo Matematico di Palermo, vol. xli (1916), p. 103; and Mr. Lidstone's Note in *J.I.A.*, vol. xlv, p. 490.

the same total of weights has been concentrated round the distance from a to $a+\lambda \leqslant b$, that is, round the larger values of $f(t)$, and each of these values is weighted with the maximum weight.

The arithmetical proof is easy. Remembering the assumed properties of $f(t)$ and $\phi(t)$, the second inequality (2) may be derived as follows:

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)\phi(t)dt \\
&\quad = \int_a^{a+\lambda} [1-\phi(t)]f(t)\,dt - \int_{a+\lambda}^b f(t)\phi(t)\,dt \\
&\quad \geqslant f(a+\lambda)\int_a^{a+\lambda} [1-\phi(t)]\,dt - \int_{a+\lambda}^b f(t)\phi(t)\,dt \\
&\quad = f(a+\lambda)\left[\lambda - \int_a^{a+\lambda} \phi(t)\,dt\right] - \int_{a+\lambda}^b f(t)\phi(t)\,dt \\
&\quad = f(a+\lambda)\int_{a+\lambda}^b \phi(t)\,dt - \int_{a+\lambda}^b f(t)\phi(t)\,dt \\
&\quad = \int_{a+\lambda}^b \phi(t)\,[f(a+\lambda)-f(t)]\,dt \geqslant 0.
\end{aligned}$$

The first inequality (2) may be derived in a precisely similar way, but is more quickly obtained by putting, in the inequality just proved,

$$\begin{aligned}
\phi(t) &= 1-\phi_1(t) \\
\lambda &= \int_a^b [1-\phi_1(t)]\,dt \\
&= b-a-\int_a^b \phi_1(t)\,dt \\
&= b-a-\lambda_1
\end{aligned}$$

Inequalities for *sums* (instead of *integrals*) may be derived by the same method. But it will repay the trouble to notice once for all, how the transition from integrals to sums may be effectuated. For this purpose we want a definition of a sum, when the limits of summation are not integers. If $x=\alpha-\theta_1$,

$y=\beta+\theta_2$, where α and β are integers, $0 \leqslant \theta_m < 1$ and $y \geqslant x-1$, we propose the convenient notation

$$\sum_x^y u(n) = \theta_1 u(\alpha-1) + \sum_\alpha^\beta u(n) + \theta_2 u(\beta+1) \quad . \quad . \quad . \quad (3)$$

it being understood that $\sum_\alpha^{\alpha-1} u(n) = 0$.

As the sum depends only on *integral* values of the argument n, we may, for the purpose of demonstrating the inequalities, put

$$u(n+\theta) = u(n) \qquad (0 \leqslant \theta < 1).$$

With this convention, (3) may be written

$$\sum_x^y u(n) = \int_x^{y+1} u(t)\,dt \qquad (y \geqslant x-1) \quad . \quad . \quad . \quad (4)$$

We have, for instance,

$$\sum_x^y 1 = y-x+1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\sum_x^y u(n) = \sum_x^t u(n) + \sum_{t+1}^y u(n) \qquad (x-1 \leqslant t \leqslant y) \quad . \quad . \quad . \quad (6)$$

From (2) we may, by (4), immediately conclude, for $y \geqslant x-1$,

$$\sum_{y-s+1}^y f(n) \leqslant \sum_x^y f(n)\phi(n) \leqslant \sum_x^{x+s-1} f(n) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where

$$s = \sum_x^y \phi(n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

provided that $f(n)$ never increases, and that $0 \leqslant \phi(n) \leqslant 1$.

If $\phi(n)=1$ or $\phi(n)=0$ or $f(n)=\text{const.}$ for all n, the two limits in (7) coincide.

It is evident that we may, in (2) and (7), allow the range of integration or summation to increase indefinitely, provided the expressions do not become meaningless.

We have assumed $0 \leqslant \phi(t) \leqslant 1$ for the sake of simplicity; but if, instead of this we have quite generally $l \leqslant \psi(t) \leqslant \mathrm{L}$, where l and L are any two different, positive or negative quantities, we need only apply the theorem to

$$\phi(t) = \frac{\psi(t)-l}{\mathrm{L}-l}$$

in order to find limits for $\int_a^b f(t)\psi(t)dt$ and for $\sum_x^y f(n)\psi(n)$. We leave the details to the reader.

The inequalities we have established are useful in mathematical analysis, their affinity to such theorems as Abel's Lemma* and the Theorem of Mean Value† being obvious. Here we are, however, only concerned with their application to actuarial problems. The possibility of this application is due to the fact, that one of the limits in (2) and (7) is, in certain cases, so close to the value of the integral or sum considered, that it may pass as a first rough approximation which becomes a very fair one on applying a simple correction.

Putting, for instance, in (2)

$$f(t)=v^t,\ \phi(t)={}_tp_x,\ a=o,\ b=\infty,\ \lambda=\bar{e}_x,$$

we find

$$\int_o^\infty v_t{}^t p_x dt < \int_o^{\bar{e}_x} v^t dt$$

or

$$\bar{a}_x < \frac{1-v^{\bar{e}_x}}{\delta} \quad \text{. . . . .} \quad (9)$$

that is the well known result that a life-annuity is smaller than an annuity-certain payable for the expectation of life.

Similarly, we find from (7), putting $x=1$, $y=\infty$, $f(n)=v^n$, $\phi(n)={}_np_x$, $s=e_x$,

$$a_x < \sum_1^{e_x} v^n \quad \text{. . . . .} \quad (10)$$

If $e_x=k+\theta$, where k is an integer, and $o \leqslant \theta < 1$, (10) may be written

$$a_x < \frac{1-v^k}{i} + \theta v^{k+1} \quad \text{. . . . . .} \quad (10\text{A})$$

It was, in fact, this particular inequality, familiar to actuaries,‡ which suggested to me the general formulas (2) and (7).

* Bromwich: An Introduction to the Theory of Infinite Series. London, 1908. P. 54 and p. 426.

† We may evidently write (2) in the form

$$\int_a^b f(t)\phi(t)dt = \int_\theta^{\theta+\lambda} f(t)dt$$

where $a \leq \theta \leq b-\lambda$.

‡ *Text-Book,* Part II, 1st Edit., p. 112.

The popular but erroneous notion that a life-annuity may be calculated as an annuity-certain payable for a duration equal to the expectation of life, yet contains an element of truth,* which may be turned to good account. The duration m of an annuity-certain, equal in value to a life-annuity, may be calculated from the equation

$$a_x = \frac{1-v^m}{i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

whence

$$m = -\frac{\log(1-ia)}{\log(1+i)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If we expand m in powers of i, remembering, that a is a function of i, and neglecting in the result powers above the first, we find the approximate formula

$$m = e_x - i\epsilon_x \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$\epsilon_x = \frac{1}{l_x}\sum_1^\infty t l_{x+t} - \frac{1}{2}e_x(e_x+1) \quad . \quad . \quad . \quad . \quad (14)$$

which may more conveniently be written

$$\epsilon_x = \frac{1}{l_x}\Sigma^2 l_{x+1} + \frac{1}{8} - \frac{1}{2}\mathring{e}_x^{\,2} \quad . \quad . \quad . \quad . \quad (14\text{A})$$

the summation being commenced from the bottom, or in *Text-Book* notation

$$\epsilon_x = \frac{\mathrm{Y}_x}{l_x} - \frac{1}{8} - \frac{1}{2}\mathring{e}_x^{\,2} \quad . \quad . \quad . \quad . \quad . \quad (14\text{B})$$

It may be worth noticing, that the function ϵ_x thus introduced has an independent meaning in actuarial science. In fact, it may be easily proved that $2\epsilon_x$ is *the mean square deviation in the estimated curtate duration of an individual life.* Supposing the life, aged x at entry, fails after t years (leaving out any fraction of a year), then the deviation from the estimated duration is $(t-e_x)$. Squaring this, multiplying by the probability, that the life will fail between ages $x+t$ and $x+t+1$, and summing for all values of t, we find the required mean square deviation, or

* The reason for this is, that we shall have $\bar{a}_x = \bar{a}_{\overline{\mathring{e}_x}|}$ provided that either $\delta = 0$, or else $\mu_x = 0$ for all values of x.

$$\sum_0^\infty (t-e_x)^2({}_tp_x - {}_{t+1}p_x)$$

$$= \sum_0^\infty (t-e_x)^2\, {}_tp_x - \sum_1^\infty (t-1-e_x)^2\, {}_tp_x$$

$$= e_x^2 + \sum_1^\infty [(t-e_x)^2 - (t-1-e_x)^2]\, {}_tp_x$$

$$= e_x^2 + \sum_1^\infty (2t - 2e_x - 1)\, {}_tp_x$$

$$= e_x^2 + 2\sum_1^\infty t\, {}_tp_x - 2e_x^2 - e_x \quad = 2\epsilon_x.$$

It is seen from (13), that e_x is really a first approximation to the true duration m. We proceed to show by a numerical investigation, that the further approximation obtained by deducting $i\epsilon_x$ from e_x produces very fair approximate values of a_x.

The experience employed for this purpose was the H^M experience (*Text-Book* graduation), and the table below gives the difference between the true and approximate values by (13) of a_x at various rates of interest. The following table of Practical Standards of Comparison reproduces the values of $\frac{1}{2}(a_x - a_{x+1})$, or approx. the variation in the annuity-value,

Differences between True and Approximate Values of a_x.

x	3%	4%	5%	6%
20	−·10	−·11	−·11	−·10
30	−·04	−·04	−·05	−·04
40	+·01	+·01	+·01	+·02
50	·03	·04	·06	·08
60	·02	·04	·06	·08
70	·02	·03	·04	·06
80	·01	·02	·02	·04

Practical Standards of Comparison.

x	3%	4%	5%	6%
20	·10	·07	·05	·04
30	·12	·09	·06	·05
40	·15	·12	·09	·07
50	·18	·15	·12	·10
60	·19	·16	·14	·12
70	·17	·15	·14	·12
80	·12	·12	·11	·10

when the age varies one half year. This is a deviation familiar to the actuary who is accustomed to calculate the "age nearest birthday", a method which is legitimate when no systematic deviation from the mean is to be feared.

Our table of differences between original and approximate values shows that these differences, when the youngest ages and highest rates of interest are left out, are small, not only in comparison with the standards, but also absolutely speaking, so that the slight systematic deviation traceable is unimportant from a practical point of view.

The best standard for measuring these deviations is, however, the *standard deviation* (or "mean error") in the value of a_x as deduced from a given experience.* The problem of calculating this standard deviation has not until comparatively recently† attracted the attention of actuaries, but as it is of importance also on other occasions, *e.g.*, when the *weights* of the ungraduated a_x are required in a direct graduation of this function, we shall occupy ourselves with the matter here. It may be remarked beforehand, that the result at which we shall arrive is practically the same as the first formula on p. 102, of G. F. Hardy's book; but the following deduction is not only easier but also safer from the theoretical point of view, making less use of approximations.

The *mean square deviation* (or the square of the mean error) is the square of the standard deviation and is denoted by μ_2. Let there be given n different and independent observations

$$\sigma_1, \sigma_2, \ldots \sigma_n$$

and an analytical function of these

$$f(\sigma_1, \sigma_2, \ldots \sigma_n).$$

Put $\sigma_r = y_r + \eta_r$, y_r being the "true" value of σ_r, and η_r the error of observation. Provided these errors are sufficiently small, f may be developed in powers of them,

* This standard deviation, has, of course, nothing to do with the standard deviation examined by Bremiker and others, in the estimated value of an annuity on a single life. The latter concerns the *application*, the former the *origin* of the table.

† G. F. Hardy: The Theory of the Construction of Tables of Mortality, &c., p. 99.

neglecting powers above the first. The coefficients of the errors of observation are, then

$$\frac{\delta f}{\delta y_1}, \frac{\delta f}{\delta y_2}, \ldots\ldots \frac{\delta f}{\delta y_n}.$$

The mean square deviation in f is, therefore*

$$\mu_2(f)=\sum_1^n\left(\frac{\delta f}{\delta y_r}\right)^2\mu_2(\sigma_r) \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In order to apply this well-known general theorem to the life-annuity, the latter must be expressed as a function of the independent q_x, that is

$$\left.\begin{aligned} a_x&=\sum_1^\infty v^t{}_tp_x \\ &=\sum_1^\infty v^t(1-q_x)(1-q_{x+1})\ldots\ldots(1-q_{x+t-1}) \end{aligned}\right\} \quad . \quad (16)$$

Now

$$\begin{aligned} \frac{\delta a_x}{\delta q_{x+r}}&=-\frac{1}{1-q_{x+r}}\sum_{r+1}^\infty v^t{}_tp_x \\ &=-\frac{1}{p_{x+r}}v^r{}_rp_xa_{x+r} \\ &=-\frac{N_{x+r}}{p_{x+r}D_x} \end{aligned}$$

whence, by (15),

$$\mu_2(a_x)=\sum_{r=0}^\infty\frac{N^2_{x+r}}{p^2_{x+r}D_x{}^2}\mu_2(q_{x+r}).$$

But, denoting as usual the Exposed to Risk by E_x,

$$\mu_2(q_x)=\frac{p_xq_x}{E_x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

therefore

$$\mu_2(a_x)=\frac{1}{D_x{}^2}\sum_x^\infty\frac{q_tN_t{}^2}{p_tE_t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This formula gives the mean square deviation in the ungraduated value of a life-annuity, and the standard deviation $\sqrt{\mu_2(a_x)}$ is, therefore, the most correct measure for the permissible deviation from the observations.

It should be remembered, that of the functions employed

* Thiele: "Theory of Observations," p 39.

in (18), D_x, p_x, q_x, and N_x are assumed to be true (or at least graduated) values, while E_x depends on the observations alone.

The calculation of $\sqrt{\mu_2}$ according to (18) is not a very laborious task. A slide-rule may be used for the purpose, the approximation obtained by this instrument being quite sufficient.

We give below specimens of the standard deviations in a_x according to various tables.

Standard deviations in a_x.

x	H^M 3%	H^M 4%	H^M 5%	H^M 6%	O^M 3%	$O^{M(5)}$ 3%	$D^{M(5)}$ $3\frac{1}{2}$%
20	·059	·048	·040	·034	·023	·065	·268
30	37	29	23	19	15	19	·063
40	38	30	25	20	15	16	58
50	44	36	31	26	16	16	69
60	52	45	40	35	18	18	82
70	67	61	55	51	20	20	99
80	·105	99	93	88	28	27	·146

They were all calculated by (18), except the figures for $O^{M(5)}$ which are G. F. Hardy's original approximations. It is seen that this test is somewhat more severe than our "Practical Standards of Comparison"; yet it does not alter the conclusion at which we had previously arrived, that the values of a_x as calculated by (11) and (13) are sufficiently accurate for most practical purposes. The standard deviations according to $D^{M(5)}$ are particularly illustrative of the liberties which may be taken with figures derived from a comparatively small experience, the total of exposed to risk according to $D^{M(5)}$ being 282,118.

It appears from (18), and has already been pointed out by G. F. Hardy, that if the Exposed for all ages are multiplied by the same constant factor k, then the standard deviation in a_x is divided by $\sqrt{k}$. The total exposed according to O^M is 7,659,454 and according to H^M 1,199,093. If the relative distribution is proportional at all ages, the standard deviation according to O^M should, therefore, be about four-tenths, of the standard deviation according to H^M, which is seen to agree very fairly except at the highest ages.

If monetary tables at a particular rate of interest i have

already been prepared, these may with advantage be taken as starting-point for the calculation of annuity-values at a different rate of interest, say i'. Putting $i'=i+h$ and

$$a'_x=\sum_1^\infty(1+i')^{-t}{}_tp_x$$

$$=\sum_1^\infty(1+hv)^{-t}v^t{}_tp_x$$

we find by (7), if $h>0$,

$$a'_x<\sum_1^{a_x}(1+hv)^{-t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

It is, therefore, natural to put, in analogy with (11)

$$a'_x=\frac{1-(1+h)^{-n}}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where it may be expected that a_x will be a first rough approximation to n. We find, in fact, developing n in powers of h and neglecting in the result powers above the first,

$$n=a_x-h\alpha_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where

$$\alpha_x=\frac{v}{D_x}\Sigma^2D_{x+1}+\frac{1}{8}-\frac{1}{2}\left(a_x+\frac{1}{2}\right)^2$$

or

$$\alpha_x=\frac{vS_x}{D_x}+\frac{1}{8}-\frac{1}{2}\left(a_x+\frac{1}{2}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

While, in (19), we must suppose $h>0$, (20) is evidently valid for positive and negative h. If h is positive, the right-hand side of (20) is $a_{\overline{n}|}$ at the rate of interest h. If h is negative and $=-k$, the right-hand side of (20) is easily seen to be $s_{\overline{n+1}|}-1$ at the rate of interest $\frac{k}{1-k}$, which may nearly always be replaced by k, as in the examples below.

For $i=0$ (20) and (21) evidently reduce to (11) and (13).

We give below for $i=3\frac{1}{2}$ per-cent, $i'=3$ per-cent and 4 per-cent, the differences Δ between the true and approximate value of a_x; further the corresponding figures when a'_x is calculated by the obvious formula

$$a'_x=a_x-h\frac{vS_x}{D_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

obtained by expanding a'_x in powers of h, neglecting powers above the first.

Differences between True and Approximate Values of a_x.

x	Δ by (20)		Δ by (23)	
	3%	4%	3%	4%
20	·02	·01	·12	·11
30	·02	·01	·09	·08
40	·01	·01	·06	·05
50	·01	·00	·03	·03
60	·00	·00	·01	·01
70	·00	·00	·01	·00
80	·00	·00	·00	·00

It should be noted that the numerical results for fractional durations were obtained by *first difference* interpolation in an interest-table (Spitzer), which is sufficient when the life-annuity is only required to two decimals. On the other hand, the value of the method in practice depends to a large extent on it being possible to content oneself with first-difference interpolation. We have, therefore, in (13) and (21) not gone beyond the first power of i, and h, though it is possible to continue these expansions.

The results by (13) and (21) seem so satisfactory, that we do not hesitate to recommend making a table of ϵ_x or α_x to three or four significant figures a regular feature of every publication of mortality-tables. The cases where isolated values of life-annuities are required may not be very frequent; still, they are sometimes wanted in transactions with life-annuities, for comparison with other tables, or for examining the effect of a change in the mortality, and the want may even increase with the facility for satisfying it.

We now proceed to show, utilizing an idea due to Mr. Meidell, how Dr. Jensen's inequalities for *convex functions* may be derived from (7).

As criterion for convex functions* we use this, that the *second divided difference*† of the function shall always‡ be $\geqslant 0$;

* Compare Galvani, *l.c.* Dr. Jensen uses as criterion

$$\psi(x) + \psi(y) \geq 2\psi\left(\frac{x+y}{2}\right)$$

and proves his theorem by generalizing this inequality.

† With regard to this notion (which goes back to Newton), *see* a Note in this *Journal* by L. Oppermann, vol. xv, p. 145.

‡ That is: for all arguments for which the function is assumed to exist.

or, if $\psi(x)$ be the function, x_1, x_2, x_3, any three different arguments, that

$$\frac{\dfrac{\psi(x_1)-\psi(x_2)}{x_1-x_2}-\dfrac{\psi(x_2)-\psi(x_3)}{x_2-x_3}}{x_1-x_3} \geqslant 0 \quad . \quad . \quad . \quad (24)$$

It follows, in particular, from (24), that if $\psi(x)$ possesses a second differential coefficient, then $\psi''(x) \geqslant 0$. This is often used as a criterion for convex functions, but is insufficient in many cases of practical importance, for instance, when dealing with an ungraduated table, or with a table graduated by mechanical or graphic methods.

For *concave* functions the sign $\geqslant$ in (24) must be replaced by $\leqslant$. We need only consider one of the cases, convexity or concavity; for if $\psi(x)$ is convex, then $-\psi(x)$ is concave, and *vice versâ*. The case where the second divided difference vanishes may be reckoned either to the convex or to the concave functions, or the expression *linear functions* may be applied.

If $\psi(n)$ is a convex function of an integral, positive or negative argument n, we have, by (24), $\Delta^2\psi(n) \geqslant 0$. Summing from n to $\beta \geqslant n$

$$-\Delta\psi(n) = \sum_n^\beta \Delta^2\psi(n) - \Delta\psi(\beta+1) \quad . \quad . \quad . \quad (25)$$

and, as $\Delta^2\psi(n) \geqslant 0$, the right-hand side of (25) is a never increasing function of n which may be called $f(n)$. Summing now again from n to β we find

$$\psi(n) = \sum_n^\beta f(n) + \psi(\beta+1) \quad . \quad . \quad . \quad . \quad . \quad (26)$$

We see from this, that a convex function of an integral argument may always be written in the form $\sum_n^\beta f(n) + \text{const.}$ where $f(n)$ is a never increasing function. Conversely, any such expression evidently represents a convex function. Similarly, the expression $\sum_\alpha^n f(n) + \text{const.}$ may be taken to represent all concave functions of an integral argument.

Now let $e(n)$ be any such function that

$$0 \leqslant \sum_n^\beta e(r) \leqslant \sum_\alpha^\beta e(r) \qquad (\alpha \leqslant n \leqslant \beta) \quad . \quad . \quad . \quad (27)$$

Assuming $\sum_{\alpha}^{\beta} e(r) > 0$, we put, for abbreviation,

$$\rho = \frac{1}{\sum_{\alpha}^{\beta} e(r)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and choose for $\phi(n)$ the function

$$\phi(n) = \rho \sum_{n}^{\beta} e(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

which, owing to (27), satisfies the condition $0 \leqslant \phi(n) \leqslant 1$.

It is easily verified, that*

$$\sum_{\alpha}^{\beta} f(n) \sum_{n}^{\beta} e(r) = \sum_{\alpha}^{\beta} e(n) \sum_{\alpha}^{n} f(r) \quad . \quad . \quad . \quad . \quad (30)$$

whence, in particular, for $f(n)=1$

$$\sum_{\alpha}^{\beta} \sum_{n}^{\beta} e(r) = \sum_{\alpha}^{\beta} (n-\alpha+1) e(n) \quad . \quad . \quad . \quad . \quad . \quad (31)$$

From the inequality proved above

$$\sum_{\alpha}^{\beta} f(n) \phi(n) \leqslant \sum_{\alpha}^{\alpha+s-1} f(n), \qquad s = \sum_{\alpha}^{\beta} \phi(n),$$

we therefore obtain by (29), (30) and (31)

$$\rho \sum_{\alpha}^{\beta} f(n) \sum_{n}^{\beta} e(r) = \rho \sum_{\alpha}^{\beta} e(n) \sum_{\alpha}^{n} f(r) \leqslant \sum_{\alpha}^{\alpha+s-1} f(n)$$

where

$$s = \rho \sum_{\alpha}^{\beta} \sum_{n}^{\beta} e(r) = \rho \sum_{\alpha}^{\beta} (n-\alpha+1) e(n) = \rho \sum_{\alpha}^{\beta} n e(n) - (\alpha - 1)$$

or, if we put $\sum_{\alpha}^{\xi} f(n) = -\psi(\xi)$,

$$\psi\left[\frac{\sum_{\alpha}^{\beta} n e(n)}{\sum_{\alpha}^{\beta} e(n)}\right] \leqslant \frac{\sum_{\alpha}^{\beta} e(n) \psi(n)}{\sum_{\alpha}^{\beta} e(n)} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The function $\psi(\xi)$ is a convex function, but not the most general convex function, for it vanishes for $\xi = \alpha - 1$, and is, according to our definition (3), linear in every interval

* Formula (30) is independent of the assumed properties of $f(n)$ and $e(n)$. This formula is, in fact, the analogon for *sums* of Dirichlet's formula for *integrals*.

$n<\xi<n+1$. But both of these restrictions may easily be removed. For it is clear, that if (32) is satisfied by a certain ψ, then it is also satisfied, if we replace ψ by $\psi+\text{const.}$ Secondly, a convex function which is not linear in the intervals $n<\xi<n+1$ cannot be larger than the function which assumes the same values for integral values of ξ but is linear in every interval of this kind.* The inequality (32) is, therefore, *a fortiori* satisfied if ψ signifies the most general convex function.

(32) is a particular case of Dr. Jensen's theorem, but is easily generalized into the complete theorem. The function $e(n)$ was arbitrary, only subject to satisfying (27); it may, therefore, be chosen so that $e(n)=0$ except at the points $n_1, n_2, \ldots n_\omega$, which have been taken in increasing order. Putting $e(n_r)=c_r$, this is evidently an arbitrary coefficient, only subject to satisfying the inequality (27) which becomes

$$0\leqslant\sum_r^\omega c_n\leqslant\sum_1^\omega c_n \qquad (1\leqslant r\leqslant\omega) \quad . \quad . \quad . \quad . \quad (33)$$

and (32) becomes

$$\psi\left[\frac{\sum_1^\omega c_r n_r}{\sum_1^\omega c_r}\right]\leqslant\frac{\sum_1^\omega c_r\psi(n_r)}{\sum_1^\omega c_r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

In this inequality n_r is still subject to being an integer (positive or negative); but this restriction also may be removed. It is, in fact, clear, that if $\psi(\xi)$ is a convex function,

* The characteristic geometrical quality of convexity, easily deducible from (24), is that the straight line connecting any two points of the curve leaves the part of the curve situated between these two points, below it. This may be proved as follows: Writing y_n for for $\psi(x_n)$, and, assuming $x_1<x_2<x_3$, (24) may be written

$$\frac{y_2-y_1}{x_2-x_1}\leqslant\frac{y_3-y_2}{x_3-x_2}$$

whence

$$y_2-y_1\leqslant(x_2-x_1)\frac{y_3-y_1}{x_3-x_1}.$$

Now, the equation to the straight line between (x_1, y_1) and (x_3, y_3) is

$$y-y_1=(x-x_1)\frac{y_3-y_1}{x_3-x_1};$$

putting, here, $x=x_2$ we see at once that $y\geqslant y_2$.

Vice versâ, (24) follows from the geometrical characteristic. This characteristic quality is also at once proved by the criterion $\psi''(x)\geqslant 0$ which is, therefore, equivalent to (24), whenever $\psi''(x)$ exists.

and $k>0$, then $\psi\left(\frac{\xi}{k}\right)$ is also convex, as follows at once from the geometrical consideration. We may, therefore, in (34) replace n_r by $\frac{n_r}{k}$. If the quantities $k_1, n_1, n_2, \ldots n_\omega$ are supposed to increase indefinitely in a suitable way, the quantities $\frac{n_r}{k}$ may be made to approach indefinitely to any previously given quantities x_r. *If, therefore,* $\psi(x)$ *is a continuous, convex function, and* x_r *never decreases, and if the quantities* c_r *satisfy the inequality* (33) *where* $\sum\limits_1^\omega c_r > 0$ then

$$\psi\left[\frac{\sum\limits_1^\omega c_r x_r}{\sum\limits_1^\omega c_r}\right] \leqslant \frac{\sum\limits_1^\omega c_r \psi(x_r)}{\sum\limits_1^\omega c_r} \quad . \quad . \quad . \quad . \quad (35)$$

This is Dr. Jensen's inequality between mean values, or rather a generalization thereof, for Dr. Jensen only proved it on the assumption that all the coefficients c_r are positive, which they need not necessarily be according to (33), although (33) is certainly satisfied if these coefficients are positive. In that case the x_r may be taken in an arbitrary order.

It should be noted, as pointed out to me by Mr. N. P. Bertelsen, that the argument employed for ψ on the left-hand side of (35), or $\frac{\sum c_r x_r}{\sum c_r}$, is not smaller than the smallest or larger than the largest of the quantities x_r. This is easily proved by going back to the inequalities (27). Summing these from $n=\alpha+1$ to $n=\beta$ we find by (31)

$$0 \leqslant \sum_{\alpha+1}^{\beta}(n-\alpha)e(n) \leqslant (\beta-\alpha)\sum_{\alpha}^{\beta}e(n)$$

or

$$\alpha\sum_{\alpha}^{\beta}e(n) \leqslant \sum_{\alpha}^{\beta}ne(n) \leqslant \beta\sum_{\alpha}^{\beta}e(n)$$

or, if $\sum\limits_{\alpha}^{\beta}e(n)>0$,

$$\alpha \leqslant \frac{\sum\limits_{\alpha}^{\beta}ne(n)}{\sum\limits_{\alpha}^{\beta}e(n)} \leqslant \beta$$

whence the result follows by proceeding as above.

From (35) the corresponding formula with *integrals* instead of *sums* is derived by putting $x_r = g\left(\frac{r}{\omega}\right)$, $c_r = k\left(\frac{r}{\omega}\right)\cdot\frac{1}{\omega}$ where $g(t)$ and $k(t)$ are integrable functions, and letting ω increase indefinitely. By the definition of an integral as a limit of a sum, we then obtain

$$\psi\left[\frac{\int_0^1 k(t)g(t)\,dt}{\int_0^1 k(t)\,dt}\right] \leqslant \frac{\int_0^1 k(t)\,\psi[g(t)]\,dt}{\int_0^1 k(t)\,dt} \quad . \quad . \quad . \quad (36)$$

where $g(t)$ never decreases, while the function $k(t)$, according to (33), must satisfy the inequality

$$0 \leqslant \int_\theta^1 k(t)\,dt \leqslant \int_0^1 k(t)\,dt \qquad (0 \leqslant \theta \leqslant 1) \quad . \quad . \quad . \quad (37)$$

It is clear, that the limits 0 and 1 in (36) and (37) may be replaced by a and b (put $t = \frac{x-a}{b-a}$ in the integrals).

The inequality (36) was proved by Dr. Jensen on the assumption that $k(t)$ is positive which, according to (37), is a sufficient although not a necessary condition.* In this case $g(t)$ need not be never decreasing.

The inequalities (35) and (36) are of a very general nature, comprising, as particular cases, a great many well-known inequalities, as pointed out by Dr. Jensen. For instance, $\log t$ being concave for $t > 0$, we find† from (35), putting $\psi(t) = \log t$, $x_r = b_r > 0$,

$$\log\frac{\Sigma c_r b_r}{\Sigma c_r} \geqslant \frac{\Sigma c_r \log b_r}{\Sigma c_r} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

or

$$\frac{c_1 b_1 + c_2 b_2 + \ldots + c_\omega b_\omega}{c_1 + c_2 + \ldots + c_\omega} \geqslant \left(b_1^{c_1} b_2^{c_2} \ldots b_\omega^{c_\omega}\right)^{\frac{1}{c_1+c_2+\ldots+c_\omega}} \quad . \quad . \quad (39)$$

If this well-known formula is applied to any other than

* The formula derived by Mr. Meidell is the particular case obtained by putting $g(t)=t$ in (36); he assumes, like Dr. Jensen, that $k(t)$ is positive, and introduces certain restrictions, necessary for the manipulation of the integrals. These restrictions are avoided by operating, as we have done in establishing (32), on *sums* instead of *integrals*. The process, by which (32) was generalized into (35), would, in the case of Mr. Meidell's formula, correspond to substituting another variable in the integral.

† Putting, in (38), $b_r = k^r$ we have (apart from the notation) inequality (A) in Mr. Lidstone's paper in *J.I.A.*, vol. xlv, p. 487.

positive c's, (33) must, of course, be satisfied, and b_r must never decrease.

Putting $c_m = S_m$, $b_m = v^{n_m}$, the inequality expresses that the *equated time* for a number of sums due at different times, as found by the usual approximate method (*Text-Book* I, pp. 24–27), *favours the debtor.**

Further, $\psi(t) = t\log t$ is convex for $t > 0$. Putting $x_r = b_r > 0$ and assuming *either* that $c_r > 0$, *or else* that (33) is satisfied and b_r never decreases, we obtain from (35)

$$\left(\frac{c_1 b_1 + c_2 b_2 + \ldots + c_\omega b_\omega}{c_1 + c_2 + \ldots + c_\omega}\right)^{c_1 b_1 + c_2 b_2 + \ldots + c_\omega b_\omega} \leqslant b_1^{c_1 b_1} b_2^{c_2 b_2} \cdots b_\omega^{c_\omega b_\omega} \quad . \quad (40)$$

a formula which has been applied in this *Journal* (vol. xl, p. 121) by Mr. S. E. Macnaghten. The logarithmic form of (40) gives, if $b_r = k^r$, Mr. Lidstone's inequality (B) in *J.I.A.*, vol. xlv, p. 487.

If $\psi(t) = t^2$ which is a convex function, we find from (35), putting $c_r = \dfrac{b_r}{x_r}$,

$$\frac{\Sigma b_r}{\Sigma \dfrac{b_r}{x_r}} \leqslant \frac{\Sigma b_r x_r}{\Sigma b_r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

or the well-known result, that the harmonical mean is not greater than the arithmetical mean. This certainly holds if b_r and x_r are positive quantities, but also provided only their quotient $c_r = \dfrac{b_r}{x_r}$ satisfies (33) and x_r never decreases.

If $\psi(t) = t^p (p > 1)$, then the function is convex for $t > 0$; putting $c_r = \dfrac{1}{\omega}$ we obtain, if $x_r > 0$

$$\left(\frac{x_1 + x_2 + \ldots + x_\omega}{\omega}\right)^p \leqslant \frac{x_1^p + x_2^p + \ldots + x^p}{\omega} \quad . \quad . \quad (42)$$

also a well-known result.†

Many other interesting examples have been given by Dr. Jensen; we content ourselves with referring to his memoir, but quote from Mr. Meidell's paper the following application to the approximate calculation of life-annuities.

* For $c_r = 1$ (39) shows that the arithmetical mean of positive quantities is never smaller than the geometrical mean.

† *See* Mr. Lidstone's paper "On the Equation of Payments, &c," *J.I.A.*, vol. xlv, p. 484. The more general inequality on the same page is obtained by putting, in (35), $\psi(t) = t^m$, $x_r = r$, $c_r = S_r$.

Considering, that the two sides of (35) approach to each other, if ψ approaches to a linear function, we may obtain an approximate formula by choosing for ψ a convex function with comparatively slight curvature.

Applying this principle to the life-annuity a'_x at rate of interest $i'+i+h$, written in the form used above

$$a'_x = \sum_1^{\infty}(1+hv)^{-t}v^t{}_tp_x \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Mr. Meidell writes

$$\psi(t) = (1+hv)^{-t}, \ c_t = v^t{}_tp_x, \ x_t = t,$$

$\psi(t)$ being evidently convex, and obtains from (35)

$$a'_x > a_x(1+hv)^{-\frac{\sum_1^{\infty} t\mathrm{D}_{x+t}}{\sum_1^{\infty} \mathrm{D}_{x+t}}} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

whence the approximate formula

$$a'_x \doteq a_x(1+hv)^{-\frac{\mathrm{S}_x}{\mathrm{N}_x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

producing a slightly too small value for a'_x.

Putting, in (45), $i=0$, and replacing thereafter h by i, we find

$$a_x = e_x v^{\frac{\Sigma \mathrm{N}'_x}{\mathrm{N}'_x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

While (46) does not, without a corrective term, produce results accurate enough for practical use, the results by (45) are very good, as appears from the following table, quoted from Mr. Meidell's paper,* where a'_x at 3 per-cent and 4 per-cent was calculated by (45), taking a_x at $3\frac{1}{2}$ per-cent, while Δ denotes the difference between the true and approximate values.

a'_x by (45). H^{M} *Text-book.*

x	3%	Δ	4%	Δ
20	22·02	·04	18·62	·04
30	19·86	·04	17·13	·03
40	17·16	·02	15·12	·02
50	13·87	·01	12·51	·01
60	10·22	·00	9·44	·01
70	6·66	·00	6·29	·00
80	3·70	·00	3·57	·00

* One or two minor errors in the last figure have been corrected.

Mr. Meidell gives other examples, but those in the table suffice to prove the practical utility of Dr. Jensen's inequality which has not yet, amongst actuaries, received the popularity it deserves.

From (36) we may, as pointed out by Mr. Meidell, derive an inequality, resembling (2) in form, but being in certain respects more, in others less general than (2). We prefer, for similar reasons as above, to operate on the sums instead of the integrals, and proceed to show how an inequality of the type (7) may be derived from (35) by Mr. Meidell's method. At the same time we shall obtain a more general result than Mr. Meidell's.

Putting, in (35), $x_r = r$, $\sigma = \sum_1^\omega c_r > 0$, $s = \sum_1^\omega r c_r$, $c_r = \frac{\phi(r)}{r}$, $\psi(\xi) = -\sum_1^\xi f(n)$, we have

$$\psi\left(\frac{s}{\sigma}\right) \leqslant \frac{1}{\sigma}\sum_1^\omega c_r \psi(r)$$

or, as $f(n)$ never increases and, consequently, $\psi(r) \leqslant -rf(r)$,

$$\psi\left(\frac{s}{\sigma}\right) \leqslant -\frac{1}{\sigma}\sum_1^\omega f(r)\phi(r)$$

or

$$\sum_1^\omega f(r)\phi(r) \leqslant \sigma \sum_1^{\frac{s}{\sigma}} f(r).$$

A lower limit for $\sum_1^\omega f(r)\phi(r)$ may be obtained in a similar way, or simply by putting

$$r = \omega + 1 - n,\ \phi(r) = \phi_1(n),\ f(r) = -f_1(n)$$

and dropping the suffixes. The result is the inequality

$$\tau \sum_{\omega+1-\frac{s}{\tau}}^{\omega} f(n) \leqslant \sum_1^\omega f(n)\phi(n) \leqslant \sigma \sum_1^{\frac{s}{\sigma}} f(n) \quad . \quad . \quad . \quad . \quad (47)$$

where

$$s = \sum_1^\omega \phi(n),\ \sigma = \sum_1^\omega \frac{\phi(n)}{n},\ \tau = \sum_1^\omega \frac{\phi(n)}{\omega + 1 - n} \quad . \quad . \quad . \quad (48)$$

it is assumed that $f(n)$ never increases, that $\sigma > 0$, $\tau > 0$, $s > 0$, and that the quantities $\frac{\phi(n)}{n}$ and $\frac{\phi(n)}{\omega + 1 - n)}$ respectively

satisfy the condition that the sum from 1 to r is not smaller than 0 or greater than σ and τ respectively.

In particular, (47) is always valid, if $f(n)$ never increases, and $\phi(n)$ is never negative.

If $\sigma=\tau=1$—a case which is evidently possible*—(47) reduces to (7), and the previous considerations have, therefore, procured a new and different test for the validity of (7). If $\sigma<1$, $\tau<1$, the limits in (47) are generally more narrow and certainly not wider than the limits in (7); if $\sigma>1$, $\tau>1$, the reverse holds. For, putting in (7) $\phi=k$, we have for any not increasing function

$$\sum_{y+1-k(y-x+1)}^{y} f(n) \leqslant k\sum_{x}^{y} f(n) \leqslant \sum_{x}^{x-1+k(y-x+1)} f(n) \qquad (0\leqslant k\leqslant 1) \quad . \quad . \quad . \quad (49)$$

If, in this formula, we put $k=\sigma<1$, $x=1$, $y=\frac{s}{\sigma}$ we have

$$\sigma\sum_{1}^{\frac{s}{\sigma}} f(n) \leqslant \sum_{1}^{s} f(n),$$

while for $k=\tau<1$, $x=\omega+1-\frac{s}{\tau}$, $y=\omega$, we find

$$\sum_{\omega-s+1}^{\omega} f(n) \leqslant \tau\sum_{\omega+1-\frac{s}{\tau}}^{\omega} f(n).$$

If, on the other hand, we put $k=\frac{1}{\sigma}<1$, $x=1$, $y=s$ resp. $k=\frac{1}{\tau}<1$, $x=\omega-s+1$, $y=\omega$, we find the two preceding inequalities with the sign $\leqslant$ changed into $\geqslant$.

From (47) we may derive a corresponding inequality for integrals, simply by putting $\phi(n)=\phi_1\left(\frac{n}{\omega}\right)$, $f(n)=f_1\left(\frac{n}{\omega}\right)\cdot\frac{1}{\omega}$, dropping the indices and allowing ω to increase indefinitely. Writing

$$\lambda=\int_0^1 \phi(t)dt,\ \mu=\int_0^1 \frac{\phi(t)}{t}dt,\ \nu=\int_0^1 \frac{\phi(t)}{1-t}dt \quad . \quad . \quad (50)$$

the result is

$$\nu\int_{1-\frac{\lambda}{\nu}}^{1} f(t)dt \leqslant \int_0^1 f(t)\phi(t)\,dt \leqslant \mu\int_0^{\frac{\lambda}{\mu}} f(t)dt \quad . \quad . \quad . \quad (51)$$

* Example: $\omega=2$, $\phi(1)=\frac{2}{3}$, $\phi(2)=\frac{2}{3}$.

it is here assumed, that $f(t)$ is a never-increasing, integrable function, that the integrals (50) exist, that $\lambda > 0$, $\mu > 0$, $\nu > 0$, and that $\phi(t)$ satisfies the condition

$$\left.\begin{aligned} 0 \leqslant \int_t^1 \frac{\phi(t)}{t} dt \leqslant \mu \\ 0 \leqslant \int_t^1 \frac{\phi(t)}{1-t} dt \leqslant \nu \end{aligned}\right\} (0 \leqslant t \leqslant 1) \quad . \quad . \quad . \quad (52)$$

The condition (52) is clearly satisfied, if $\phi(t)$ is never negative. The limits 0 and 1 in (50)–(52) may evidently be replaced by a and b.

Comparison between (51) and (2) leads to similar results as the comparison made above between (47) and (7); we need not go into details.

Comparison between (51) and (47) shows, that it is preferable to operate on sums instead of integrals; for, in (51), we assumed the existence of the integrals (50) which puts certain inconvenient restrictions on the $\phi(t)$, for instance, that it must vanish at the limits of integration.

The result proved by Mr. Meidell was (51), but on the assumption that $\phi(t)$ is positive which, according to (52), is no absolute condition.

Returning to Dr. Jensen's inequalities, it is clear that the particular case obtained by putting $x_r = r$ in (35), or $g(t) = t$ in (36), is the most important for the actuarial applications. The formulas are for this purpose conveniently written

$$\left.\begin{aligned} \sum_1^\omega c_r \psi(r) \geqslant \psi(\xi) \sum_1^\omega c_r, \\ \xi = \frac{\sum_1^\omega r c_r}{\sum_1^\omega c_r} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (53)$$

which is identical with (32), and

$$\left.\begin{aligned} \int_a^b k(t)\psi(t)dt \geqslant \psi(\eta) \int_a^b k(t)dt \\ \eta = \frac{\int_a^b tk(t)dt}{\int_a^b k(t)dt} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (54)$$

We repeat that a *sufficient* criterion for the applicability of these inequalities is, that c_r and $k(t)$, respectively, are positive, and $\Delta^2\psi(r)$ and $\psi''(t)$ respectively $\geqslant 0$.

Many of the functions employed in actuarial calculations satisfy these conditions, at least when Makeham's hypothesis, holds. Amongst the functions which are always or usually convex for adult ages we quote μ_x, q_x, P_x, $\frac{1}{a_x}$, N_x, S_x, R_x, while l_x, d_x, D_x, C_x, M_x should be treated with more caution.

In order to use the limits, produced by (53) and (54), with advantage, ξ and η must be easier to calculate than the expression for which we seek a lower limit. If the curvature of ψ is only slight, we may, as in Mr. Meidell's example, obtain an approximate formula by replacing the sign $\geqslant$ by $=$. But it may be added that in this case the condition, that ψ should be convex, becomes superfluous, and that the equations then express the "First Theorem of Mean Value" with the important addition that ξ and η are known approximately. It may further be added that if we only want an approximate formula and not an inequality, the whole of the previous argument may be spared and replaced by the simple consideration, that the obvious *identity*

$$\Sigma c_r(a+br)=(a+b\xi)\Sigma c_r \quad . \quad . \quad . \quad . \quad (55)$$

becomes an *approximation,* if the linear function $(a+bx)$ is replaced by any function with a slight curvature. In this form the theorem was already given some years ago by Mr. Lidstone* who further showed how the approximation ξ may sometimes be improved. If we put

$$\Sigma c_r F(r)=F(\zeta)\Sigma c_r \quad . \quad . \quad . \quad . \quad . \quad (56)$$

and, for abbreviation,

$$\xi=\frac{\Sigma rc_r}{\Sigma c_r},\ \xi'=\frac{\Sigma r^2c_r}{\Sigma c_r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

then Mr. Lidstone finds approximately

$$\zeta=\xi+(\xi'-\xi^2)\frac{F''(\xi)}{2F'(\xi)} \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Other formulas of a similar nature may be devised. We confine ourselves to one that produces exact results if applied

* *J.I.A.*, vol. xlv, p. 483.

to a polynomial of the second degree. Let $F(t)$ be such a polynomial; then

$$F_1(t) \equiv \frac{F(t) - F(0)}{t}$$

will be of the first degree, and (55) may be applied to $\Sigma c'_r F_1(r)$ where $c'_r = rc_r$. We find

$$\Sigma c'_r F_1(r) = F_1(\xi_1) \Sigma c'_r, \quad \xi_1 = \frac{\Sigma rc'_r}{\Sigma c'_r}$$

or, introducing $F(t)$ and c_r,

$$\Sigma c_r F(r) = \left[F(0) + \xi \cdot \frac{F(\xi_1) - F(0)}{\xi_1} \right] \Sigma c_r \quad . \quad . \quad . \quad (59)$$

where

$$\xi = \frac{\Sigma rc_r}{\Sigma c_r}, \quad \xi_1 = \frac{\Sigma r^2 c_r}{\Sigma rc_r} \quad . \quad . \quad . \quad . \quad . \quad (60)$$

This, being exact for a polynomial of the second degree, may be used as an approximation, if $F(t)$ can approximately be represented by a polynomial of the second degree.

On the other hand there are cases where (58) produces better results than (59); it seems, therefore, that further investigations are desirable as to the circumstances under which these methods of approximation are efficient.

Dr. Jensen's theorem may be expresssed very simply in the language of probabilities. Assume, that the coefficients c_r are all positive, and put $p_r = \dfrac{c_r}{\sum\limits_1^\omega c_r}$ whence $\sum\limits_1^\omega p_r = 1$. Let $x_1, x_2, \ldots x_\omega$ be all the possible values of x, and let p_r be the probability of the occurrence of x_r. Then, by definition, $\sum\limits_1^\omega p_r x_r$ is the "mathematical expectation of x", and Dr. Jensen's theorem may be expressed thus:

A continuous convex function of a mathematical expectation is not greater than the mathematical expectation of the function.

If the mathematical expectation is denoted by the symbol E, then the inequality (35) may be neatly written thus

$$\psi(Ex) \leqslant E\psi(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Taking for instance, $\psi(t) = t^2$, we find $(Ex)^2 \leqslant E(x^2)$; as a

matter of fact $E(x^2)-(Ex)^2$ is the square of the mean error and, therefore, positive.

We shall not continue this subject further; but doubtless (61) must have many applications in the theory of probabilities where it is often more important to be able to indicate a limit than to find a very precise one.*

* Compare a paper by Mr. Meidell on the theory of limitation of risk in *Transactions* of the Seventh Actuarial Congress, vol. i, p. 88.

ACTUARIAL NOTE.

Claim Acceleration Reserve for Endowment Assurances.

1. If death claims are, on the average, payable $\left(\frac{1}{2}-k\right)$ of a year after the moment of death, the claim acceleration reserve will be $ki\,.\,{}_tV^1_{x\overline{n}|}$ for a unit policy effected at age x, payable at $x+n$, and valued on the assumption that the $(t+1)$th annual premium is due and unpaid. This assumption will simplify the formulæ and is sufficiently accurate when a relatively small adjustment is in question.

2. Dropping the factor ki (which in any given case is a constant that can be applied to totals), and representing the sum assured by S and bonus additions by B, we have for the value of the adjustment:

$$(S+B)A^1_{x+t:\overline{n-t}|}-S\frac{A^1_{x\overline{n}|}}{\mathbf{a}_{x\overline{n}|}}\mathbf{a}_{x+t\,\overline{n-t}|}$$

or, writing A_0 for $A^1_{x\overline{n}|}$ and A_t for $A^1_{x+t:\overline{n-t}|}$

$$(S+B)A_t-S\frac{A_0}{\mathbf{a}_{x\overline{n}|}}\mathbf{a}_{x+t\,\overline{n-t}|}$$

$$=SA_0-S\frac{A_0}{\mathbf{a}_{x\overline{n}|}}\mathbf{a}_{x+t\,\overline{n-t}|}+BA_0+(S+B)\left[A_t-A_0\right]$$

$$=A_0\left[S\,{}_tV_{x\overline{n}|}+B\right]+(S+B)\left[A_t-A_0\right]$$

3. The following table gives specimen values of A on the basis $O^{M(5)}$ 3 per-cent. It will be seen from this table

(i) That for any given value of n, $\Delta_x A$ increases approxi-

mately in G.P. This might have been anticipated, since it is known that $\Delta_x a_{\overline{xn}|}$ and $\Delta_x(\Delta_n a_{\overline{xn}})$ behave in this way, and $A = 1 - d(1 + a_{\overline{xn-1}|}) - \Delta_n a_{\overline{xn-1}|}$, so that $\Delta_x A = -\Delta_x[a_{\overline{xn-1}|} + \Delta_n \,.\, a_{\overline{xn-1}|}]$.

(ii) That for any given value of the maturity-age, $M = x + n$, the values of A change but slowly in the most important part of the table, but they naturally fall off as the assurance nears maturity.

$O^{M(5)}$ 3 *per-cent Values of* $A^1_{\overline{xn}|}$.

x	$(x+n)$					
	40	45	50	55	60	65
20	·104	·127	·149	·174	·200	·226
25	·089	·116	·143	·172	·203	·236
30	·069	·100	·133	·168	·205	·245
35	·040	·079	·118	·160	·205	·253
40	...	·047	·096	·147	·201	·259
45	...	...	·059	·122	·188	·259
50	...	...	...	·077	·159	·246
55	...	...	...	...	·104	·215
60	...	...	...	...	...	·145

4. From (i) it follows that a mean value of A_0 for any given value of $n-t$ (the unexpired term) can be found by means of the average maturity-age derived by the Z method. Over the whole E.A. business, the average value of M is usually rather under 60, so that if we take age 60 we should be fairly accurate with a small error on the side of stringency. A glance at the table shows that ·2 should be a fair average value of A_0, and that $[A_t - A_0]$ is generally negative but not important except at the smallest unexpired terms. Neglecting $[A_t - A_0]$, which process is on the side of stringency, we have the result that the adjustment is

$$< \cdot 2ki[S\,_tV_{\overline{xn}|} + B]^*$$

If $k = \frac{4}{12}$, the claim acceleration-reserve on a 3 per-cent

* If the adjustment were taken to be $A_0 ki[S_tV_{\overline{xn}}$ + *value* of Bonus], the omitted terms would be $S(A_t - A_0) + B(A_t - A_0 A_{x+t\overline{n-t}})$. The first term is generally positive, but the second may in many cases be negative, and it is not easy to show that on the whole the adjustment so calculated would be on the safe side.

basis is therefore less than $\cdot 002 \times$ [the total endowment assurance reserve $-$value of Bonus$+$actual amount of Bonus]. This result amply confirms a remark made by Mr. J. M. Warden, F.F.A. (T.F.A., vol. vii, p. 348) to the effect that his experience had satisfied him that an adjustment of $\frac{1}{2}$ per-cent on the reserve "is larger than is necessary when the average period is two months." Similarly, Mr. H. W. Brown said (*loc. cit.* p. 346): "I think a reserve of ·2 of 1 per- "cent of the liability in respect of bonus additions and a "reserve of ·15 per-cent of the remaining liability—that is "the liability for the sum assured—will be found to give a "sufficiently stringent reserve for early payment of claims "under the endowment assurance class."

5. The whole adjustment found by the above or any more closely accurate process is so small that probably most actuaries will think it desirable to increase it in practice. A very good way to do this is to throw out altogether the adjustment in the value of the premiums and to take the claim acceleration reserve as A_0ki [total value of sums assured and bonus].

6. It will be observed from the approximation that the average value of A_0 (on which alone depends the adjustment in the value of the premiums) should be a fairly close one. The principal error in the final result arises from the neglect of the term $[S+B][A_t-A_0]$ which affects the sum assured and bonus only. Hence, if it is desired to apply the adjustment separately to the value of the sum assured and bonus and to the value of the pure premiums, the former adjustment will be A_0ki [total amount of sums assured and bonus] which will be considerably in excess; and we can also find a close approximation to the adjustment in the value of the premiums, namely,

$$-A_0ki\Sigma S\mathbf{a}_{x+t\,\overline{n-t|}}/\mathbf{a}_{x\overline{n|}} = -A_0ki\Sigma S\mathbf{a}_{x+t\,\overline{n-t|}}(P_{x\overline{n|}}+d) =$$

$-A_0ki$ [Σ value of pure premiums $+ \Sigma$ sums assured $- \Sigma$ value of sum assured]. This assumes that the values of the sums assured are known independently of the values of the bonus additions. If, however, the sums assured and bonuses are valued together the values of the sums assured alone can be found with sufficient accuracy for the present purpose by valuing the bonus additions in broad groups and deducting the value so found from the value of the sums assured and bonuses combined.

7. Similarly, the correction to the net premium *income* is

$$-A_0ki\Sigma\frac{S}{\mathbf{a}_{\overline{xn}|}}=-A_0ki\Sigma S(P_{\overline{xn}|}+d)$$

which is easily calculated since $\Sigma SP_{\overline{xn}|}$ and ΣS are known.

8. In the above Notes the $O^{M(5)}$ Table has been used for convenience, but the results on the O^M Table would be similar. The case of a combined O^M and $O^{M(5)}$ Valuation would be more complicated, but it does not appear probable that the results could be substantially different, and there seems little objection to using a single table in calculating the value of a small adjustment, especially as the factor A_0ki found from the single table is actually applied to the larger reserves resulting from the combined tables.

9. It is not claimed that the method here outlined does more than give an easily-applied method of calculating the adjustment on the safe side: but the adjustment is itself so small that such a method is all that is usually required.

G. J. L.

REVIEWS.

Standard Mortality Ratios incident to Variations in Height and Weight among Men. Report of the Joint Committee of the Actuarial Society of America and the Association of Life Insurance Medical Directors.

[New York, 1918.]

This Report—which is apparently intended to be followed by similar reports dealing with other factors affecting the eligibility of lives for assurance—is the outcome of a suggestion that the work of the Medico-Actuarial Investigation should be supplemented by the definite recommendation, for general use, of "standards by which to measure the value of lives for insurance." It presents, in a series of tables, a revised and analyzed statement of the results already published by the Medico-Actuarial Investigation Committee in regard to the influence of build and excessive abdominal girth on mortality.

No useful purpose would be served by discussing the methods employed in the construction of the tables or the meaning of the results. Nor, indeed, is the material available for such a discussion, for after referring to three methods of dealing with the preponderance of the experience of early policy years and stating that certain further statistics had been furnished by the Mutual of New York the Report proceeds: "With this additional information

"and with such further data as could be obtained from the experience "of other individual companies, the Committee prepared tables of "percentages in practicable form for use in the selection of risks "which seemed to it to be fairly representative of past experience "after giving due weight to the results obtained under the three "plans referred to above." It is sufficient, in view of the form that the results take, that we have here the authoritative recommendation of a Committee of experts who have obviously given careful consideration to the various points involved. It is immaterial that other methods might have produced somewhat different results, because a mortality ratio is in any case a blend of more or less unknown constitution.

The important point about the Report is that it seems to mark the definite adoption of the mortality ratio by American actuaries as a practical representation of the mortality of a class of lives. This appears to us rather a confession of failure from the research point of view. The mortality ratio tells us in effect that, according to a particular experience, the actual deaths among a group of lives having certain defined characteristics were a specified percentage of the expected according to a given standard. That is to say, it gives the value of $\sum_0^n q'_{[x]+n}\mathrm{E}_{[x]+n}/\sum_0^n q_{[x]+n}\mathrm{E}_{[x]+n}$—obviously a most complicated fact to translate into an extra premium. It tells us nothing as to the incidence of the mortality, nothing as to the relative values of q' and q. We fully recognize the impossibility of having a separate mortality table for, *e.g.*, 30-lb. overweights of medium height and aged 30-34 at date of assurance; but between this and the mortality ratio 1·18 there are numerous possibilities. It seems to us that it would be more illuminating, even if it should involve a broader grouping, to have (say) the values of $\Sigma q'_{[x]+n}\mathrm{E}_{[x]+n}/\Sigma\mathrm{E}_{[x]+n} - \Sigma q_{[x]+n}\mathrm{E}_{[x]+n}/\Sigma\mathrm{E}_{x+n}$ for the first 5 years of assurance and for subsequent durations. The mortality ratio may, perhaps, owe its official adoption to the vogue of the numerical method of rating, by which one arrives at the percentage by which the "total value of a risk" exceeds or falls short of the normal by adding together the several percentages by which its mortality ratios exceed or fall short of the standard in such respects as build, family history, pulse rate, occupation, &c. We doubt, however, whether this is sufficient to commend it. Without insisting unduly on the fact that, as Mr. Macaulay in particular has shown (T.A.S.A., vol. xii, p. 290), the method leads to absurd results in extreme cases—for this might be true of a method which nevertheless had a scientific basis in average cases—we cannot find that so far any valid argument for it has been adduced, except perhaps the argument from experience. It has been suggested that it is a more methodical substitute for the old-fashioned plan of mentally balancing *pros* and *cons*, and that by its means the errors "due to the exercise of personal judgment on a combination of factors" are greatly reduced. It would seem, however, that even with the numerical method the impossibility of dividing a risk into absolutely independent factors

must involve liability to error. It is obvious, for example, that it would be fallacious to obtain the value of a lightweight with a tubercular history by adding together the separate mortality ratios for light weight and personal history of tuberculosis.

A second point suggested by the Report is, that the adaptation of an Experience to the purpose for which it is intended to be used appears to be increasingly regarded in America as part of the regular procedure of a mortality investigation. The principle is as old as the employment of "amounts" as the basis of a mortality table, but it is now being freely applied to the data included in an investigation. The most remarkable example of its exploitation is afforded by the recently-compiled American and Canadian Experience, in which the data have been disciplined by the restriction of the amount observed on any one life to $100,000 and by the limitation of the contributions of certain large companies. It is a principle which affords scope for much ingenuity. In the Report under notice, for example, one of the methods of dealing with the preponderance of early policy years' experience was to reduce the actual and expected deaths of any policy year "in proportion to the number of years of issue entering therein." But even after this adjustment the several durations must have entered into the aggregate with varying weights, owing to the varying amounts of new business in different years. It is a question whether it would not be more satisfactory, in forming aggregate tables, to combine the various durations in definite and known proportions.

Simple Interest Tables (£ *s. d.*). *By* William Schooling, C.B.E., F.R.A.S.

[Pp. xvi + 172. London: Sir Isaac Pitman & Sons, Ltd. 21*s.* net.]

Of the 170 pages of tables contained in this work, 50 are taken up by a table of $\log 240in/365$ (for $i=1$ to 6 per-cent by eighths, $6\frac{1}{4}$ to 7 per-cent by fourths, 8, 9 and 10 per-cent and $n=1$ to 369), and 100 more by a table of $\log S$ for $S=1$ to 23999. It will be apparent that the effect of entering the first table with a specified rate of interest and a specified number of days, of entering the second with a specified number of pounds, of adding the results, and finally of entering the second table inversely with the sum thus obtained, will be to give (to the nearest penny and within the limits of the table) the simple interest (in pence), for the specified rate and number of days, on the specified amount. The second table is arranged with £S and the equivalent in £ *s. d.* of S pence as alternative arguments—or, rather, as argument and result—on the left and right respectively of the log, so that the computer can take out the antilog directly in £ *s. d.* (whence the title of the tables) instead of having to convert the pence; and there are supplementary tables for dealing with amounts outside the range of the tables. Moreover, with the object of rendering the tables capable of being used by a computer who

does not understand logarithms, the tabulated values—5-figure logs with characteristics—are printed without decimal points, and with 10 added when (as in parts of the first table) they would be negative. Mr. Schooling's idea, as set forth in the preface, is that we can use logs without understanding their nature just as we can switch on electric light without knowing how it is produced, or turn on water without understanding a pump. In the particular case under consideration this may be true as regards calculations falling within the range of the tables, but our feeling is that if a clerk who knew nothing about logarithms were asked to calculate by these tables interest on an odd amount (not an integral number of pounds), or interest amounting to more than £99 19*s.* 11*d.*, anything might happen. And anyone with a knowledge of logarithms would, we think, find it more convenient to use a table of log $in/365$—*i.e.*, a table similar to Mr. Schooling's first table, but without the pence factor, in conjunction with an ordinary log table.

Any tables for the purpose of calculating simple interest for broken periods naturally challenge comparison with John Laurie's well-known work. Since the publication, in 1831, of the 1st edition (of which "Every genuine Copy" bore the author's signature), this work has become such an established institution that five years ago it reached—with comparatively little material change—a 42nd edition. It seems almost useless to contend with such a work. There is obviously something in its monumental simplicity that appeals to the human mind—something that makes it almost futile to suggest that Sellar's Tables (on similar general lines, but with a logically reduced range of amounts and with the results given to the nearest tenths of a penny) are far more serviceable. As compared with either Laurie or Sellar, Mr. Schooling gives, of course, a far more extensive range of rates. But most of these rates are very seldom wanted, and for the occasional calculation at an odd rate there are devices familiar to all computers of simple interest, or, at the worst, there is the despised multiplication. And there is, to our mind, the fatal objection to Mr. Schooling's process that it requires three openings of the tables as against Laurie's or Sellar's one.

Mr. Schooling's fertility in ideas on the subject of the calculation of simple interest is not, however, exhausted by the £ *s.* *d.* tables. In the preface to these tables he states that he has in preparation tables on an alternative plan, and a specimen is given on pp. 4–7 of the present work. This is simply a table of $S\,in/365$ for $S = 100$, $200\ .\ .\ .\ 900$, $i = \cdot 01$ and $n = 1\ .\ .\ .\ 185$ with a few additional values—the results being given to 6 decimal places. It pushes Sellar's logical reduction of amounts to the limit rendered possible by the use of decimals, and it requires only one opening. Although involving in most cases more simple addition than Laurie, a table of this kind is a simple and reasonable instrument which will commend itself to every computer who is used to working in decimals. Whether it will compete successfully with John Laurie is another matter.

CORRESPONDENCE.

THE RELATION BETWEEN THE RATES OF MORTALITY, OR DECREMENT, FROM PARTICULAR CAUSES, AND THE RATES FROM ALL CAUSES, WITH SOME REMARKS UPON THE PREVALENT MISUSE OF THE TERM "RATE."

To the Editors of the Journal of the Institute of Actuaries.

DEAR SIRS,—With reference to Mr. D. S. Savory's letter *J.I.A.*, vol. li, p. 65, regarding the mortality due to the war, I beg to submit the following remarks dealing with the subject somewhat generally.

The distinction between a probability and a rate is one that deserves to be more clearly emphasized. The term "rate of mortality", as generally employed, might be defined as meaning the proportion dying in a year out of a number exposed to the risk of death for the duration of a year or until death within the year, the words "per annum" being understood. This particular definition has most conveniently led to the function q_x in an ordinary mortality table becoming invested with a dual capacity; for q_x not only represents the annual rate of mortality as above defined, but also the probability of dying within a year, at age x.

When, however, we come to split q_x into its component parts, the distinction is a real one. For we can subdivide q_x, the total probability of dying in a year, into the partial, mutually exclusive, and additive probabilities of dying in a year from cancer, consumption, &c., whose sum equals exactly the total probability of dying in a year; but we cannot so split up the rate of mortality q_x into partial rates of mortality, for the rates of mortality from different diseases are proportions of dissimilar things and are therefore not additive, *e.g.*, the rate of mortality from consumption is the proportion dying from consumption of those exposed for a year to that particular risk, while the rate of mortality from cancer is a proportion of another thing.

Coming now to the particular problem (which is somewhat similar to that dealt with by Dr. Sprague in *J.I.A.*, vol. xxi, p. 406, *see* also Ackland, vol. xxxiii, p. 194): Given the values q_{ng}, the rate of mortality from normal and war causes combined, and q_n, the rate of mortality due to normal causes, it is desired to find an expression for the rate of mortality due to the war.

Let d_n and d_g be the number dying during a war year from normal and war causes respectively, out of l persons alive at the commencement of the year; then $\frac{d_n}{l}$ and $\frac{d_g}{l}$ will be the probabilities of dying from the respective causes mentioned. Let these probabilities be designated q'_n and q'_g, as distinguished from q_n and q_g, the corresponding rates of mortality.

Then
$$q_n = \frac{d_n}{l - \frac{1}{2} d_g}$$

Dividing both numerator and denominator by l, we have

$$q_n = \frac{q'_n}{1 - \frac{1}{2} q'_g}$$

Whence
$$q'_n = q_n\left(1 - \frac{1}{2} q'_g\right) \quad \text{. . . . . A}$$

Similarly
$$q_g = \frac{d_g}{l - \frac{1}{2} d_n}$$

Whence
$$q'_g = q_g\left(1 - \frac{1}{2} q'_n\right) \quad \text{. . . . . B}$$

It is easily seen that
$$q_{ng} (\text{or } q'_{ng}) = q'_n + q'_g \quad \text{. . . . . . C}$$

Substituting first A and then B, we derive from C

$$q'_g = \frac{q_{ng} - q_n}{1 - \frac{1}{2} q_n} \quad \text{. . . . . D}$$

$$q'_n = \frac{q_{ng} - q_g}{1 - \frac{1}{2} q_g} \quad \text{. . . . . E}$$

From B,
$$q_g = \frac{2q'_g}{2 - q'_n};$$

whence, substituting for q'_g and q'_n the values given in D and E,

$$q_g = \frac{2(q_{ng} - q_n)}{1 - \frac{1}{2} q_n} \times \frac{1 - \frac{1}{2} q_g}{2 - q_{ng}},$$

i.e.,
$$q_g\left\{\left(1 - \frac{1}{2} q_n\right)\left(2 - q_{ng}\right) + \left(q_{ng} - q_n\right)\right\} = 2\left(q_{ng} - q_n\right)$$

Whence we get
$$q_g = \frac{q_{ng} - q_n}{1 - q_n + \frac{1}{4} q_n q_{ng}} \quad \text{. . . . . F}$$

Equations D and F represent respectively the probability of dying and the rate of mortality, from war causes, as expressed in terms of the known quantities q_{ng} and q_n. Formula F agrees with that got by Prof. Cantelli except for the third term in the denominator, his result being $\frac{q_{ng}-q_n}{1-q_n}$. The Professor's reasoning as quoted by Mr. Savory evidently omits certain considerations.

With regard to the statement credited to the Professor that his result follows from Karup's theorem

$$p_{ng}=p_n \times p_g$$

the probabilities p_n and p_g in this particular instance are not altogether subjects for multiplication as they do not relate to events that are quite independent. The true relation is

$$p_{ng}=(1-q_{ng})=1-(q'_n+q'_g)$$

Formulas D and E would furnish the means of expressing p_{ng} in terms of p_n and p_g.

As regards the method adopted by Prof. Hersch, this would have been quite correct had he been dealing with probabilities instead of rates, for $q'_g=q_{ng}-q'_n$, but as indicated above, it is not accurate to add or subtract rates.

Among our writers there is, so far as terminology is concerned, a want of precision in discriminating between a rate and a probability. In a pension fund for instance, we require the *probabilities of exit* by death, withdrawal, &c. Out of a number who attain a given age we require to know simply the proportion that will go off the fund by each of the modes of exit within a year, and the proportion remaining on the fund at the end of the year; it is a simple splitting up of the given number, the results representing the mutually exclusive probabilities in respect of each of the possible ways in which the event can happen, and adding to unity. These proportions or probabilities are commonly and loosely described as rates of mortality, withdrawal, &c., but they are clearly not properly so described. Another term used with an appearance of greater precision is "the rate of mortality while on the active list", but this is also inexact. The rate of mortality while on the active list is in fact correctly represented by the following expression:

$$\left.\begin{array}{r}\text{Rate of mortality while}\\ \text{on the active list}\end{array}\right\}=\frac{q'_x}{1-\frac{1}{2}(wq'_x+rq'_x)}$$

where q', wq' and rq' represent the probabilities of exit by death, withdrawal, and retirement. For if not, what is the correct designation of this expression?

The cause of the confusion is evident. Insurance is effected

against a probability, never against a rate; and when dealing with ordinary life insurance, actuaries could hardly help seeking for the probability of dying within a year, or the simple proportion of those dying in a year out of a number alive at the commencement. It was expedient to define the abstract function known as the rate of mortality so that it would coincide with this, rather than to define it in any other way, such for instance as that proposed by Dr. Farr; and therefore when we are dealing with a single force such as death or withdrawal, the probability and the rate have the same value. In short, actuaries obtained the probability and called it the rate. The term "rate", however, which a nice discrimination would have restricted to the abstract idea, was allowed to impose itself everywhere, even in respect of probabilities with quite a different value, to the extent that the idea of a probability with its resultant simplicity appears to have been banished to a precarious footing on the margin of consciousness.

It is curious to note how even some of the most eminent of our writers have paid homage to the usurper. The late Mr. Manly, for instance, obtained what were apparently true rates of mortality and withdrawal (q_x and wq_x), and used them as probabilities of exit, admittedly as an approximate measure, but without an appropriate change in designation, following in this respect the prevalent custom (*J.I.A.*, vol. xxxvi, pp. 211, 260, 261). The retention of the incorrect terminology is responsible for half the difficulties alluded to on p. 4 of vol. xlii, besides rendering the reasoning obscure, and leaving its mark on the formulas there given. On the other hand, in vol. xxxix, p. 133, Mr. George King lays down very clearly the correct procedure for deriving the probabilities of exit required in a pension fund, but his ultimate retention of the incorrect term "rate" has led to his explanation being unnecessarily cumbered by the fiction (doing violence to the facts) that "in getting out the rate of mortality we must therefore treat the withdrawals and retirements as at risk for the whole year", &c.: a relative, apparently, of the older fiction that in getting out the rate of mortality simpliciter, deaths were to be treated as at risk for the whole year.

In Insurance functions of all kinds it is always *the probability* that we require, and the expulsion of the term "rate" herefrom at once removes the accompanying obscurities and rationalizations, and clears the channel of thought.

A great deal would be gained if a special symbol were used, where appropriate, to designate a probability as distinct from a rate (as in this communication for example), and if at the same time the concrete functions employed in financial computations, particularly those used in respect of pension funds, were given their proper designations. This need not in any way interfere with the undoubted convenience gained from the use of the term "rate" in the purely general sense. We could very easily speak of the general characteristics of the rates of mortality, &c., for example, and at the same time take care to describe our algebraic and tabular functions as probabilities or proportions.

I should add that the above examples are not quoted by way of criticism of the eminent writers named but merely as illustrations.

I am, Dear Sirs,

Yours faithfully,

A. T. TRAVERSI.

Wellington,
New Zealand.
10 *August* 1918.

P.S.—Though in the problem dealt with by him Dr. Sprague very clearly differentiated between a rate and a probability, nevertheless his awkward expression "the annual marriage rate among bachelors who do not die in the year" (*J.I.A.*, vol. xxi, pp. 413 and 415) involves some confusion regarding the definition of a rate, and it is surprising that his terminology in this instance should apparently have remained unchallenged. If "rate" is defined as in the second paragraph of this letter (*mutatis mutandis*), it is clear that the function in question could be described with the most rigid accuracy and with greater simplicity as *the annual marriage rate among bachelors*, notwithstanding Dr. Sprague's deliberate rejection of the latter expression in favour of his own.

Dr. Sprague's phrase is in fact a distinct misdescription of the function, and a realization of this fact would be of considerable help to students. If we were really in pursuit of the marriage rate among bachelors who do not die in the year we should require to exclude altogether from the figures the bachelors who die in the year. True, it might at first sight appear that there should be no difference between the annual marriage rate among bachelors and that among bachelors who do not die in the year, seeing that the rate among bachelors who die in the year is nil: but as indicated above, we cannot add or subtract rates.

Similar remarks apply to the phrase "the annual death rate among bachelors who do not marry in the year."

A. T. T.

THE NATIONALITY OF TETENS.

To the Editors of the Journal of the Institute of Actuaries.

Dear Sirs,—May I be permitted, as a Corresponding Member of the Institute, to call attention to an excusable error concerning one of my countrymen, which I happened to notice in that reliable standard work, the Institute *Text-Book.* I find that the first inventor of commutation-columns (J. N. Tetens) is called a German professor,

although the place where he was born, namely, Kiel, was Danish at that time and belonged to Denmark for the next 127 years. I need not go into details, as these have already been given by F. Hendriks in the first volume of this *Journal* (p. 2); only I may add, that the error has probably arisen from the fact that Tetens, although professor at a Danish University, published scientific papers in German, in order to obtain a wider circulation for them.

Yours very truly,

J. F. STEFFENSEN.

Ehlersvej 8,
Hellerup, Denmark.
28 *March* 1919.

THE INSTITUTE OF ACTUARIES.

At the Council Meeting held on 10 December 1918, the following Address to the King was ordered to be signed and sealed:

TO THE KING'S MOST EXCELLENT MAJESTY.

MOST GRACIOUS SOVEREIGN,

We, your Majesty's dutiful subjects, the President, Council and Members of the Institute of Actuaries, respectfully beg leave to present to your Majesty our loyal congratulations on the success achieved by your Majesty's Naval, Military and Air Forces in the present War, on the signing of the Armistice, and on the prospect of a Peace which will establish more firmly than ever the authority of the Empire in the Counsels of the World, and the position of your Majesty and the Royal House in the affections of the people.

We trust that our special knowledge and experience may be found of use in the work of reconstruction, in which we are prepared to assist your Majesty's Government to the utmost of our power.

We desire to tender to your Majesty the assurance of our heartfelt loyalty, and to offer our humble and grateful thanks for all that your Majesty and our gracious Queen have done during the War, by personal effort and example, and by never-failing sympathy with all classes, to maintain the courage and resolution of the people, and to promote that spirit of unity and self-sacrifice which has done so much to secure for your Majesty's Forces the success which has at length rewarded their heroic efforts and devotion to duty.

We fervently pray that your Majesty may long rule in peace and happiness over a prosperous and contented Empire.

Given under the Common Seal of the Institute at Staple Inn Hall, this Tenth day of December, in the year of our Lord, One thousand nine hundred and eighteen.

L.S.

Geoffrey Marks, *President.*

A. Levine,
W. Palin Elderton, } *Honorary Secretaries.*

The following resolution was unanimously passed at the Ordinary General Meeting of the Institute held on 16 December 1918:

The Members of the Institute of Actuaries in General Meeting assembled, resolve:

> That fraternal greetings and congratulations on the conclusion of the War be offered to the Actuarial Societies in France, Belgium, Italy, and the United States of America. They feel that the friendly relations which have always existed between the Members of the Actuarial profession in the British Empire and in the Allied countries have been rendered even more cordial by the common sacrifices made in the great struggle for liberty and justice now so happily ended. They have every confidence that the spirit of comradeship brought forth in the recent struggle will endure and will prove of lasting value in the work to be undertaken by the Members of the Profession in the coming days of Reconstruction.

Obituary.

FREDERICK LIGHTON BRISTOW, Probationer of the Institute, Lance Corporal, 7th London Regiment.
Killed in Action 20 *March* 1916.

JOHN WILLIAM EWART ALEXANDER, Probationer of the Institute, 2nd Lieutenant, 10th Battalion, Norfolk Regiment.
Died on Service 14 *April* 1916.

VINCENT JOHN AUSTIN, Probationer of the Institute, Private, 5th City of London Regiment.
Killed in Action 1 *July* 1916.

RAYMOND COLE, Probationer of the Institute, Private, 16th Middlesex Regiment.
Killed in Action 1 *July* 1916.

THOMAS MIDDLETON, B.A., Student of the Institute, Captain, 97th Brigade, Highland Light Infantry.
Killed in Action 1 *July* 1916.

CHARLES SIDNEY SHILSON, Probationer of the Institute, Private, 5th City of London Regiment.
Killed in Action 1 *July* 1916.

MALCOLM HOWARD GRIGG, Probationer of the Institute, 2nd Lieutenant, 26th Battalion, Manchester Regiment.
Killed in Action 9 *July* 1916.

HARRY WALTER BROWN, Probationer of the Institute, Private, Royal Naval Division.
Died on Service 23 *January* 1917.

THOMAS GRAHAM CUNLIFFE, Probationer of the Institute, Private, 20th Battalion, Royal Fusiliers.
Killed in Action — *May* 1917.

JOHN CROMWELL HURLEY, Associate of the Institute, Lance Corporal, 22nd Battalion, Australian Imperial Force.
Killed in Action 3 *May* 1917.

RALPH GEORGE GALE, Probationer of the Institute, 2nd Lieutenant, 8th Battalion, South Staffordshire Regiment.
Killed in Action 12 *October* 1917.

WILFRED BRADLEY, Probationer of the Institute, Private, Honourable Artillery Company.
Died of Wounds 6 *November* 1917.

Cyril Percy Maddox, Probationer of the Institute, Private, 16th County of London Regiment.

Killed in Action 20 *November* 1917.

James Henry Orr, Probationer of the Institute, Captain, 210 Siege Battery, Royal Garrison Artillery.

Killed in Action 30 *November* 1917.

George Frederick Thomas Ascott, Probationer of the Institute, Private, London Rifle Brigade.

Killed in Action 25 *March* 1918.

Frederick Defries, Associate of the Institute, Captain, 5th Battalion, Middlesex Regiment.

Killed in Action 6 *April* 1918.

Henry Jackson Mills, Student of the Institute, Lieutenant, 19th Battalion, Machine Gun Corps.

Killed in Action 30 *May* 1918.

Alfred Richard Harriss, Probationer of the Institute, Private, 1/15th London Regiment.

Died of Wounds 13 *September* 1918.

Charles George Gifkins, Probationer of the Institute, Gunner, 139 Heavy Battery, Royal Garrison Artillery.

Killed in Action 15 *October* 1918.

Edward Albert Newland, Student of the Institute, 2nd Lieutenant, 24th Battalion, Royal Fusiliers.

Killed in Action 23 *October* 1918.

Donald Arthur Roberts, Probationer of the Institute, Flight Lieutenant, Royal Air Force (attached H.M.S. "Iron Duke").

Died on Service 30 *January* 1919.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Group Insurance. *By* P. H. McCormack, A.I.A., *of the Provident Mutual Life Assurance Association.*

[Read before the Institute, 28 April 1919.]

THE President, in his recent address, referred to the rapid growth of group insurance in the United States, and suggested that the system was well worth the study of members of the actuarial profession in this country. The following notes may possibly be useful as an introduction to the subject. Most of the facts have been collected from contributions to American journals, a list of which is given at the end of the paper.

It has often been realized that all the lives in certain categories might be insured in bulk, selection against the insurance company being thereby avoided without the necessity for medical examination. An attempt to put this principle into practical operation has been made by means of a system devised in America under which the employees of a firm may be insured by a single policy without medical examination. This system is known as group insurance. Although there are various modifications of the plan, a group policy is generally renewable from year to year on a term basis, the cost of the premiums being borne by the employer. The sums payable at death in individual cases are usually commensurate with annual wages. The only qualifications necessary for insurance under the scheme

are active service and membership of an eligible group; and it is claimed that the scope is so wide as to include the unfit and those beyond the usual age limits. Incidentally, group insurance also covers workers who, through poverty or indifference, fail to make any personal provision for their dependents.

Although a policy bearing some resemblance to the present system was issued nearly twenty years ago, it was not until July 1912 that the first group insurance contract of note was effected in the modern form. Since that date the scheme has attained great popularity in the United States, having been adopted by many of the most important employers of labour in the country, including railroads, shipbuilding companies, and banks. As mentioned by the President in his address, one firm alone has insured its employees numbering half a million on the group system. It is reasonable, therefore, to suppose that the method offers some advantages over ordinary or industrial life insurance. Ordinary life insurance, subject to annual, half yearly, or quarterly premiums, is of course beyond the means of the majority of wage-earners, while industrial insurance, on which the premiums are collected weekly at a relatively high cost, is often only sufficient to meet the funeral and other expenses which are incurred immediately on death. One of the objects of group insurance is to meet this difficulty by enabling many grades of wage-earners, who could not otherwise be adequately insured, to obtain the benefit of substantial life insurance protection, at least during the period of their employment.

The Principles of Group Insurance.

The distinctive feature of group insurance is that the unit of insurance is not the single life but a group of lives. As the group is the unit of measurement, the individual members need not all be standard lives, and no medical examination is required; for it is only necessary to consider the eligibility of the group as a whole. If, however, a group is to be eligible for insurance, it obviously must not contain an undue proportion of damaged lives. In order to protect the insurance company against adverse selection in this respect, it is essential that the group should exist for some purpose other than that of insurance; and, moreover, membership of the group should in itself be to some extent an index of good health and activity. These conditions

may be fulfilled by a group consisting of the employees of one firm, provided all or substantially all are included. The insurance may sometimes be restricted to certain departments, or to male employees only, and casual labourers are generally excluded. With a view to the prevention of adverse selection on the part of an entire group containing an undue proportion of impaired lives, a minimum limit of 100 lives is usually fixed in America for group insurance without medical examination. If the number of lives is less than 100, the group is not necessarily ineligible for insurance, but a modified form of medical examination would generally be required, the degree of stringency of the examination depending on the size of the group. In such cases it might of course be necessary to exclude impaired lives.

Another important safeguard against the exercise of adverse selection by the members of a group is that there should be no individual option as to the amount of insurance. That is to say, the sum insured must either be a flat amount or follow a definite formula.

Group insurance has sometimes been criticized as unsound because there is no medical examination as a general rule. But when life insurance business was started, medical examination was not required, and schemes of insurance without examination have been offered by English companies in recent years. Medical examination is only a means to an end; but it is not necessarily the only means. Its object is to maintain the standard of mortality and prevent selection against the office. As is well known, the effect of medical selection for all practical purposes wears off after a few years, the select lives merging into the general body, which may be regarded as a sample population. The group insurance system under adequate safeguards automatically gives sample populations, in which there is no adverse selection. Such groups are therefore eligible for insurance, provided there is no undue occupational hazard. But it is very important that no opportunity for adverse selection against the insurance company should be given; and for this reason the system should not be applied to clubs or societies connected with a particular trade or employment where the membership is voluntary. An interesting example of what seems to be a misapplication of the group insurance principle may be mentioned. The Federal Insurance Department, while apparently taking no objection to group insurance, has expressed the

opinion that the system is not at present legal in Canada. Certain Canadian municipalities, however, during the war have insured the lives of soldiers recruited for over-seas service with American Companies on a group basis, the premiums being paid by the municipalities. Bodies of soldiers, although good lives from a medical point of view, are not suitable for group insurance owing to the catastrophe hazard which exists in any particular regiment on active service. In other words, the risks are not sufficiently decentralized to allow of the operation of the law of average. In considering the eligibility of a group for insurance, occupational hazard is an important factor. The wages-staff of a brewery company, for example, would hardly form a suitable risk for group insurance, except, possibly, in the case of an insurance company doing a very large business and having special experience in dealing with hazardous risks. Again, two firms might carry on business of a similar character, but it does not follow that both would be accepted for group insurance on the same basis, as the general conditions of employment in the two cases might be altogether different. American companies, before accepting a proposal for group insurance and determining the scale of premiums to be charged, usually make a general inspection of the premises, similar to that made in connection with workmen's compensation insurance. Careful attention is paid to such matters as sanitation, lighting and heating, fire and accident prevention devices, class of workers employed, and conditions of labour, as well as to any special risks incidental to the employment. Companies which transact workmen's compensation business have the advantage of already possessing the necessary organization for making these inspections.

The group method may be adopted in connection with any kind of insurance, whether term, whole life, endowment, or annuity, and policies on all these plans have been issued; but group insurance, as it exists in America at the present time, is practically restricted to a modified form of term insurance, under which the employer pays the whole of the premium. As a general rule, each employee is insured, so long as he remains in the service of his employer, for one, two, or three years' salary, with say a minimum of \$500 and a maximum of \$3,000, which may in some cases be increased, subject to medical examination. Occasionally the insurance is for a flat amount, which may or may not increase with length of service. Sometimes past

service is also taken into account. A group insurance policy does not necessarily cover employees who remain in the firm's service after having passed the general age for retirement; but the system is very elastic, and members who are above the usual ages for ordinary life insurance are often included. New employees are admitted to the scheme, though there is usually a probationary period of three to twelve months' active and continuous employment, which is considered sufficient to protect the insurance company against errors of judgment on the part of the employer in engaging unfit workers. This regulation also excludes casual employees who only remain with the firm for a short time. Unless precautions are taken, there is a risk of bad lives exercising adverse selection by obtaining employment with a firm which has adopted the group insurance system. Some companies, therefore, reserve the right to require medical examination of new entrants to existing groups, but may be prepared to waive this right if they are satisfied that the methods adopted by the firm in selecting new employees are sufficiently stringent.

As the insurance is on a yearly renewable term basis, the premium in individual cases naturally increases from year to year. But this is not necessarily a disadvantage, as the employer has only to consider the total premium payable, which in an average group remains fairly constant, owing to the fact that the increased cost of insuring continuing members of the staff is more or less counterbalanced by the lower cost of the young lives who replace retiring members. In a progressive business the average premium may decrease for some years.

The payment of premiums by the employer avoids the main danger of assessmentism, which arises from the free choice possessed by individual members as to whether they will remain in the society or not and the assumption that a sufficient influx of new lives will be maintained. When the supply fails, the premium charges rapidly increase, with the result that the good lives withdraw and the company becomes insolvent. Under the group system, however, this opportunity for individual selection does not exist, as the insurance must either continue as a whole or lapse as a whole. But the financial position of the employer is a factor which may affect both the insurance company and the employees; for in the event of the firm tending towards insolvency the first lives to leave its service would in the ordinary course be the young and healthy. The loss of the young lives

would not necessarily have any serious effect on the insurance company, as the policy would lapse unless the firm were able to meet the increased premium charge. If, however, owing to the withdrawal of healthy members, the firm were left with an excessive proportion of impaired lives, the group would no longer be eligible for insurance, although there might be no direct selection against the office. But as it is the practice of most American companies to guarantee the renewal of group policies for a term of years varying from five to twenty on the original premium basis, it is possible that the insurance company might be compelled to renew the policy, in which case it would probably incur a loss on the particular group until the expiry of the guaranteed term or until the firm in financial difficulties could no longer pay the premium. As, however, the unit of insurance under this system is the group, and not the individual life, any loss on one group would be averaged with the profit and loss on others; for it is not contemplated that every group must necessarily show a profit. Moreover, insolvency, or a tendency thereto, would be unlikely to occur in more than a small proportion of the insured firms, and the special circumstances mentioned would therefore seldom arise. But this possibility should serve as a warning against undue latitude in granting the option of renewal for a long term, especially in the case of companies whose group insurance business is on a small scale. The term for which rates are guaranteed has become a competitive feature with some American companies, but this is evidently an unfortunate development.

Policy Conditions.

Group insurance contracts are issued in the form of a single "blanket" policy, granted to the employer for the benefit of his staff and covering all members included in the scheme. The insurance is usually restricted at the outset to members who are in active employment. The employees are not named individually in the policy, but the necessary particulars of the insured and their beneficiaries, with the amount of insurance in each case, are recorded in a subsidiary schedule, card index, or book. Each employee receives a certificate, supplied by the insurance company, stating the terms and conditions of the scheme, the sum insured and other individual details being filled in by the employer.

Under the renewable term plan the policy is usually effected

for twelve months, but may be renewed from year to year, like a workmen's compensation policy. As already mentioned, an option is often given to the employer to continue the insurance for a term of years on the same premium scale as that adopted at the outset and scheduled in the policy. Occasionally the contract is binding on both sides for a limited period.

Employees remain insured so long as they continue in the active service of the firm ; but absence for a limited time, whether due to sickness or not, does not invalidate the insurance. Sometimes, however, this privilege is conditional on payment by the employer of the whole or part salary during the period of absence. An employee leaving the service during the currency of a group policy is usually given the option within thirty-one days of effecting a whole life or endowment insurance without medical examination, for an amount not exceeding that for which he was insured under the group policy, at the ordinary tabular premium corresponding to his attained age. In a subsequent section of the paper this option is discussed from the point of view of the insurance company. The provision is compulsory in some states for all group insurance policies, the main object being, no doubt, to safeguard the employee against losing the benefit of his insurance when it is most needed, as he would do on a strictly term basis in the event of prolonged ill-health. In connection with state laws relating to group insurance, it is interesting to note that in New York group policies covering at least 100 lives are excluded from the limitation on new business which was imposed some years ago.

If the sum insured varies according to remuneration, the amount is increased, subject to a maximum, whenever an advance in wages takes place. Alterations in the amounts of insurance and the inclusion under the scheme of eligible new employees may be automatic or may take effect upon notice by the employer to the company. Group insurance policies are free from restrictions regarding occupation and travel, but war risks and those resulting from riots or civil commotion would probably be excluded. There is no suicide clause, since moral hazard is assumed to be eliminated by the methods of group selection. In the event of understatement of age, the claim is usually settled in full, and the premium payment adjusted accordingly.

The business is generally conducted on a non-profit basis ; but some companies, charging higher premium rates, grant a

dividend or refund of a proportion of the profits each year. For this purpose the insured groups are arranged in a few classes according to the degree of occupational hazard involved, and, the profits for each class having been ascertained, separate rates of dividend are declared. The method is of course a little more complicated than the ordinary system, but it possesses the following advantages :

1. It avoids any necessity for refinement in premium rating of various risks ;
2. It provides a premium margin to meet over-average mortality ;
3. It prevents dissatisfaction which might arise under the non-profit system if the premium payments largely exceeded the claims ;
4. It minimizes the risk of loss through the guarantee of the original premium scale for a term of years.

Premium Basis.

The basis adopted for group insurance premiums in the early days of the business was the American Experience Table with interest at $3\frac{1}{2}$ per-cent. This method, however, gives excessively high premiums at low ages, and the rates now charged for young lives are often smaller than the net premiums, although at the older ages increasing loadings are necessary. Since the American $3\frac{1}{2}$ per-cent forms the legal standard of valuation, companies charging lower rates than the net premiums according to this table are required in some states to maintain a "deficiency reserve" equal to the commuted value of the amount by which the premium charged falls short of the statutory net premium. The following specimen rates of premium charged by one American company for five classes of group insurance

Rates per 1,000.

Age	American $3\frac{1}{2}$% Net Rates	$O^{M(5)}$ $q_x \times 1{,}000$	English Life No. 5 (Males) $q_x \times 1{,}000$	Insurance Commissioners' Table (Males) $q_x \times 1{,}000$	(1)	(2)	(3)	(4)	(5)
25	7·79	6·89	6·36	4·15	6·01	6·26	6·76	7·51	8·51
35	8·65	8·37	10·19	6·44	6·41	6·66	7·16	7·91	8·91
45	10·79	12·00	15·63	11·20	9·17	9·42	9·92	10·67	11·67
55	17·94	20·83	26·02	21·87	18·87	19·12	19·62	20·37	21·37
65	38·77	42·21	50·57	46·48	45·88	46·13	46·63	47·38	48·38

risks are taken from a paper read by Mr. H. Pierson Hammond at a meeting of the National Convention of Insurance Commissioners at St. Paul in August 1917. The American Experience $3\frac{1}{2}$ per-cent net one year term rates and the values of q_x according to the $O^{M(5)}$ Table and the English Life No. 5 and National Insurance Commissioners' Tables (Males) are added for comparison.

As indicated by the above scales, a limited degree of extra risk arising from occupational hazard may sometimes be covered by uniform additions to the premiums; but considerable refinement in rating, such as occurs in workmen's compensation business, is not usually attempted in group insurance. Occasionally, however, an eligible risk may be offered for which special rates may have to be devised. Although the individual members may not all be exposed to the same occupational hazard, any extra premium required is applied to the whole group as the unit of insurance. Since the premiums are paid by the employer, no hardship is caused by this method.

When a proposal for group insurance is received and the premium basis agreed upon, the insurance company must be supplied with a census of employees, including details of the remuneration at each age, in order that the exact amount of initial premium may be calculated. As, however, the preparation of this statement may involve some delay, the risk is often accepted on payment by the employer of a provisional premium of from one to one-and-a-half per-cent of the total annual wages. An adjustment will probably be necessary when full information is available, but in an average group the required premium usually lies within the limits mentioned. Further adjustments in the premium, consequent upon staff changes, frequently take effect at monthly intervals, although the actual cash settlement may only be made once a year. But sometimes the premiums are paid monthly, in which case the rates may be true monthly premiums.

Since the sum insured is usually a function of the annual wages, an alternative method of rating, analogous to that adopted in workmen's compensation business, would be to fix the premium as a percentage of the payroll, with a modification in respect of maximum and minimum limits of insurance. In determining this percentage, the general age distribution of the employees would of course be taken into account, although a fresh calculation every year based on the exact amount insured

at each age would not be required, as in the case of the usual method of computing the premium. But a census of the staff would be necessary at the outset, and also at intervals during the currency of the policy in order that the percentage might be altered in the event of an appreciable change in age distribution. It is therefore doubtful whether the alternative method would on the whole be simpler than the usual plan; and, moreover, it would represent a considerable departure from the established principles of life insurance practice.

In May last two American actuaries, Messrs. Cammack and Morris, published an investigation of the joint mortality experience during the years 1913 to 1917 of the Aetna and Travelers Insurance Companies in respect of group insurance policies. An interesting feature of the experience is the stability of the business; for out of 949 policies issued in the five years there were no more than 30 terminations, and of these only 21 were cases of actual discontinuance of insurance. There were in all 310,911 years of exposure and 2,434 deaths, the corresponding amount exposed being $222,026,715 and the amount terminated by death $2,137,700. High mortality was experienced in the case of electric and street railroads, underground and elevated railways, electric light and power companies, steel foundries, and leather industries, the mortality in tanneries being especially unfavourable. On the other hand, the furniture, woodworking, and textile trades showed favourable mortality, probably due to a relatively small proportion of fatal accidents. The results confirm the view that there is a minimum degree of selection against the office when the whole premium is paid by the employer and all employees are covered. Some years ago group insurance policies were issued to mutual aid associations of police forces, railroad employees, &c., but these show heavy selection against the insurance companies.

The experience was compiled on a calendar year basis, the exposure in year "0" comprising the fractional year's exposure in the year in which a policy was issued. The mortality in policy year "0" is very light, but it afterwards increases considerably, although no regular progression is shown. This increase in the rate of mortality is, no doubt, largely a result of the continued insurance of members who are no longer actively employed. Excluding the experience in extra-hazardous occupations, the actual deaths in policy years one to four for all ages combined were 88 per-cent by numbers and 90 per-cent

by amounts of the expected according to the Medico-Actuarial Table. The experience is very favourable at the younger and older ages, but between 40 and 65 the mortality exceeds the expected. The authors suggest that this is partly due to a strong tendency of middle-aged employees to understate their ages. The rates charged should therefore be sufficiently high to cover these mis-statements. The expected deaths were also calculated by the American Table, but this represents the experience still less closely than the Medico-Actuarial Table. The latter table is evidently more suitable than the American Experience as a basis of group insurance premiums, and some companies have adopted it for that purpose.

Although the Aetna and Travelers experience is immature and not of great volume, it is useful as a measure of mortality during the early years of group insurance. But future mortality may be appreciably affected by changes in labour turnover and by an increased exercise of the option given to employees of continuing their insurance on a permanent basis after leaving service. An early experience should therefore be used with caution as a criterion of rates to be charged in the future; for considerable loss might result from any general cutting-down of premiums.

Social Aspects of Group Insurance.

The principal contingencies which may arise during employment, and against which it is possible to make provision, are death from accident, disablement, premature death from natural causes, sickness, and loss of earning capacity through old age. Statutory compensation is awarded to an employee or his dependents in the event of disablement or death by accident in the course of his employment, though there is no legal liability in respect of the other contingencies mentioned. Group life insurance represents a development of the principle underlying workmen's compensation insurance, since it enables an employer to provide for the moral responsibility arising in cases of premature death among members of his staff. In combination with a pension scheme, the system affords to employees a considerable extension of the compensation benefits secured by law. In some cases it takes a more comprehensive form than a mere term insurance, although the latter is the usual plan. For instance, a group insurance policy effected by a New York bank included the following benefits:

(*a*) Insurance of one year's salary in the event of premature death;

(*b*) In cases of total disablement from any cause, 100 per-cent of salary to be payable for the first month, 80 per-cent for the succeeding eleven months, and 60 per-cent thereafter during continued disablement;

(*c*) A pension commencing at age 65 of an amount equal to 2 per-cent of the aggregate salary received during continuous service with the bank.

Such policies are rare, but they may perhaps foreshadow a future development of the system which will increase its importance as a social factor.

One of the advantages claimed for group insurance is that it tends to create harmony between the employer and employed, and any system which improves the relations between capital and labour has a special value at the present time. But this advantage cannot be secured unless the whole premium is paid by the employer; for if a part of the premium has to be contributed by the employees, the rule must apply to those members who are not in favour of the scheme equally with those who approve of it, since it would be contrary to the principles of group selection to allow any individual option. The difficulty would therefore not be avoided by making the adoption of the scheme conditional on the consent of a majority of the employees. The term insurance method is particularly unsuitable for a contribution system on account of the increase in the premium as the age advances and the temporary nature of the insurance benefit, which is not altogether overcome by the option sometimes given of taking up an ordinary policy on leaving service. The automatic termination of the insurance at the retiring age would naturally cause great discontent on the part of an employee who had paid contributions during the whole of his working life. If, however, the entire premiums are paid by the employer, this grievance does not arise, especially if a pension begins when the insurance lapses. Although it is possible to provide for the pension as well as the insurance by adopting the group system on an endowment insurance basis, the cost of such an arrangement would generally be prohibitive, especially in the case of firms with a high labour turnover. The term insurance basis gives the necessary cover at a minimum cost to the employer, and no question arises as to surrender values in the case of employees who leave the firm.

In the opinion of many important American employers, group insurance tends to prevent workers from drifting away through mere unrest; for, in the absence of a substantial reason for leaving, an employee would probably hesitate before giving up his insurance by joining a firm which did not adopt the system. If, therefore, group insurance reduces the labour turnover, it may save considerable expenses to the employer, as the cost of selecting and training new employees is often very high. No corresponding advantage would result from a small increase in wages equivalent to the group insurance premium.

Group insurance is economical in practice, the expenses of management being lower than in ordinary life business for the following reasons:

1. No medical fees are paid;
2. Individual solicitation is avoided, and the cost of new business thereby reduced;
3. The commission payable to agents is on a lower scale than in the case of ordinary policies. Much of the business is obtained direct;
4. The issue of a single policy to cover a group of lives saves clerical labour. In the book-keeping system, the figures for each group are carried forward from year to year *in toto*, the adjustments resulting from staff changes being kept up to date by a continuous method.

Some of the large firms which have effected group insurance policies with American companies could, of course, undertake their own insurances. There are, however, two difficulties in this. In the first place, it would probably be necessary to set up a specially organized insurance department; and, in the second place, if any dispute arose in connection with a claim, it might be made an occasion of discontent among the employees, whereas if the matter were in the hands of an independent insurance company the firm could not be held responsible for the decision.

Insurance of British Wage-earners.

Insurance provision on a small scale is already made by a considerable proportion of the wage-earners of this country by means of industrial policies. Although the individual amounts are low, the aggregate volume of business transacted is very large, total funds of nearly sixty-five millions sterling having been held by British companies in respect of these policies on

31 December 1916. But the industrial system, while serving a useful purpose in placing the benefits of life insurance within easy reach of the working-class population, unfortunately involves an economic waste on account of the heavy cost of making the weekly collections from door to door, which are an essential feature of the system. The result is that the premiums payable by industrial policy-holders are considerably higher in proportion to the sum insured than those payable by ordinary policyholders, and it is therefore very difficult for the majority of wage-earners to make adequate provision against death and old age.

Attempts have sometimes been made by associations of employees to dispense with industrial policies by grafting life insurance benefits on existing pension funds, generally without any actuarial advice; but as the rates of contribution are rarely adequate, this plan in most cases leads to insolvency.

In view of these difficulties, it has often been predicted that group insurance would be widely adopted in England after the war; but although the business has already been commenced, it seems very doubtful whether it will ever become general in this country. English firms, when adopting a scheme of staff life insurance, frequently stipulate that their own contribution shall be conditional upon the payment of an equal amount by the employees included in the scheme; but, for reasons already stated, the group insurance system in its usual form does not lend itself to this arrangement. It would, of course, be possible for the firm to defray the whole cost of insuring the staff on the short term plan, and to collect a contribution from them to be applied in purchasing a more permanent form of insurance, such as a whole life policy with premiums ceasing at the retiring age. But the difficulty would be to fix the amount of the member's contribution, having regard to the fact that in individual cases the firm's contribution increases every year. The principle of equal contributions would obviously have to be abandoned, as it would not be practicable to increase an employee's contribution so long as his salary remained the same. If the firm's contributions, as well as those paid by the employees, were used to purchase limited payment whole life policies, one of the chief objects of group insurance would be defeated, because the wage-earners would no longer obtain the substantial life insurance protection which is possible under the term system at a relatively low cost.

The privilege granted to employees of effecting an ordinary policy without medical examination places group insurance on a convertible term basis, but only in a restricted sense, as the option is limited to employees who leave the service. In the exercise of the option a maximum degree of selection against the office may be expected, as in the majority of cases it is unlikely that a healthy man taking up employment with another firm would wish to avail himself of the option, even if he had the means to do so. As group insurance is still in an early stage, it is probable that the full effect of this selection has not yet been felt; but the whole question of the option is evidently one of the most important points arising in connection with the business. The concession has been justified on the following grounds:

1. The group insurance premiums may be specially loaded to allow for the option;
2. The permanent premium payable is the full tabular rate, medical fee and commission being saved;
3. In a large number of cases the cost of an ordinary policy is prohibitive, and the option is therefore not exercised.

But unless the possibility of a considerable degree of adverse selection is taken into account, it would seem unwise to grant the option. An alternative plan, limiting the company's liability in this respect, would be to issue the group insurance policy on a two-year term basis instead of on a one-year term basis, no permanent policy being offered on leaving service. Under the one-year plan it might be necessary to exclude from the next year's insurance a member who happened to be ill at the end of the first or any subsequent year: a condition which would sometimes cause great hardship. The two-year plan obviates this difficulty to a considerable degree, and the cost is only slightly more than under the one-year plan. In order to obtain the full benefit of the option of renewal contained in the two-year plan, this option would only be exercised in the case of members whose state of health rendered them ineligible to continue the insurance on a two-year basis. The method would, of course, be inadequate in the event of prolonged ill-health.

Messrs. Burn and Symmons, in referring to the group system (*J.I.A.*, vol. xlix, pp. 231–232), suggested that it would be illegal in this country to issue a life insurance policy which did not contain the names of the insured, and moreover that legal difficulties might arise with regard to insurable interest,

although the employer obtained no direct benefit under the policy. Possibly, however, these difficulties might be overcome by issuing the contract on an indemnity basis, and not as a policy of life insurance.

Group insurance is not intended to take the place of ordinary life insurance, but only to supplement it by making provision against premature death. It is said that the adoption of a group scheme has often resulted in a greater appreciation of the value of life insurance in general, and has thereby assisted ordinary business. But it is conceivable that temporary insurance under a group system might sometimes be relied upon in cases where a more complete life policy would otherwise be effected.

Although group insurance possesses several advantages, it seems to leave many of the problems connected with the insurance of wage-earners to a large extent unsolved.

The following is a list of some of the principal papers on the subject:

Group Insurance.—By V. R. Smith. (*Proceedings* of the Insurance Institute of Toronto, 1913–1914).

Group Insurance.—By W. J. Graham. (*Transactions* of the Actuarial Society of America, October 1916).

Group Insurance.—By R. B. Trousdale. (*Annals* of the American Academy of Political and Social Science, March 1917).

Group Life Insurance and its Possible Development.—By E. B. Morris. (*Proceedings* of the Casualty Actuarial and Statistical Society of America, April 1917).

Life Insurance in Groups (1912–1917).—By H. Pierson Hammond. (National Convention of Insurance Commissioners, St. Paul, August 1917).

Joint Mortality Experience of the Aetna Life and Travelers Insurance Companies on Group Policies.—By E. E. Cammack and E. B. Morris. (*Transactions* of the Actuarial Society of America, May 1918).

Abstract of the Discussion.

Mr. F. P. SYMMONS congratulated the author on the clear and logical way in which he had dealt with a form of assurance

that might bulk very largely in the public eye in the near future. Before the war pension and sickness funds and benefit funds of a similar nature were on the increase. The labour position since the war tended to accelerate that increase, and he hoped that benefit funds would help in the solution of the difficult problems with which they were faced. It was evident that group insurance could be granted at what were apparently cheap rates. The cost of collection was negligible, and expenses of administration were comparatively small. Adverse selection was a matter which depended on the care with which the groups were selected, and the *bonâ fides* of the employer. In addition, group insurance was said to form a valuable means of introducing other business to the companies, and Mr. Day, President of the Equitable Life Assurance Society of the United States, in an address delivered in December 1913, remarked that group insurance was a supplement to, rather than a substitute for, other forms of insurance, and it operated not to replace but to create a demand for individual assurance. As to how far these anticipations had been or would be fulfilled, there was no evidence. From the employer's point of view, it was urged that the scheme operated as an inducement to workers to remain with one firm rather than to sacrifice the benefits of the assurance by transferring their services elsewhere. If, however, the practice of group insurance became to any extent widespread, the value of this advantage to employers must considerably decrease, for if all employers adopted schemes of this sort no sacrifice would be involved in changing from one firm to another. The cost to the employer, however, was inconsiderable, amounting, roughly, to 1 per-cent of the pay roll, taking, for example, the case where one year's salary was assured. With the average pension fund the minimum contribution might be taken as 5 per-cent, and frequently 6, 7 or 8 per-cent was paid. From the employee's point of view, he thought it fair to say he received cover which was limited to the period that he remained in sufficiently good health to continue in that particular occupation. The value of this cover could be tested by the rate of mortality experienced while on the active list by the members of any ordinary pension fund. In all the pension funds with which he had had any connection the rate of mortality on the active list could be fairly represented at the earlier ages by the $O^{[M]}$ rates for the first year of assurance. From forty-five upwards, perhaps, the second year rates applied. The $O^{[M]}$ rate of mortality for the first year at age twenty-five was 2·8 per thousand, while the lowest group insurance premium quoted in the paper for that age was 6·01 per thousand. At age thirty-five the $O^{[M]}$ rate was 3·6, as against 6·41; at age forty-five (for the second year of assurance), 7·75, as against 9·17; and at fifty-five 13·29, as against 18·87. If the mortality experience of pension funds were reproduced by group insurances—and the schemes as regarded mortality would appear to be analogous—the premiums charged provided adequate margins, unless they included a charge for the option of continuance. From the employee's point of view, all he received was little more than an accident insurance. When his

health declined to the extent that he was unable to continue in his employment, the cover from the group policy ceased.

In the third paragraph of the paper it was stated that group insurance gave the wage-earners adequate cover, whereas ordinary life assurance, subject to annual, half-yearly and quarterly premiums, was beyond the means of most manual workers, whilst industrial assurance with weekly collections entailed a relatively high cost and only provided sufficient to meet the funeral and other expenses incurred at death. Surely the word "adequate", if applicable at all, was only so to a most limited degree. The amount assured might be adequate, but when it was remembered that the probability of its becoming payable was small, owing to factors apart altogether from rates of mortality, the adequacy of the cover was seen to be to a great extent illusory. He did not overlook the fact that an endeavour was apparently made to meet the objection that cover ceased just when most needed, by granting the option to effect an insurance under any of the tables of the issuing company, irrespective of the state of health of the life in question. If, however, as supposed by the author, the majority of the wage-earners could not afford ordinary insurance when young and in employment, it was evident that when advanced in years and unemployed, or only employed at lower rates of wages—as they would in all probability be just at the time they wished to exercise the option—it would be difficult for them to take advantage of that option and effect an insurance under any ordinary table. It was stated in the paper that the granting of the option had been justified on three grounds, one of which was that the cost of an ordinary policy was prohibitive and the option was therefore not exercised. This might be a factor which the companies were justified in taking into account, but it showed also that the option failed to give group insurance the value of a permanent contract of life assurance, which could, practically speaking, only be cancelled at the will of the assured. He very heartily supported the author's conclusion that, although group insurance possessed some advantages, it seemed to leave many of the problems connected with the insurance of wage-earners to a large extent unsolved.

Mr. W. SCHOOLING, C.B.E., said that as a visitor he would like to express his gratitude to the author for an extremely interesting paper, on a subject which was of far-reaching importance in connection with labour and social questions. Perhaps the most useful contribution he could make to the discussion was to point out the possibility of meeting an objection to which group insurance, as commonly understood, was open. It was contended that a system of insurance of this kind made for good relations between employers and employees, and for continuity of service; that continued service was of greater value than discontinuous service; and that consequently, if group insurance produced continuous service, expenditure upon it by the employer might be an economy and not an expense. Mr. Symmons, however, made the very natural criticism on this contention that, if group insurance became popular

and was very extensively adopted, it took away this advantage. The possibility of making fairly sure of retaining continued service would be attained if there were contributions to permanent benefits, as distinct from merely the temporary insurance protection for, say, one year's service, which was the general feature. If, for example, the employer and the employed accumulated savings regularly, and if, instead of letting the group insurance policies be for one year's salary, the term policies were for the difference between a specified sum, which might be one year's salary, and the savings accumulated by the employer and the employed, that would really be converting the whole thing into endowment assurance, with, perhaps, the investment part of the business done by the employers and the employees more or less on their own account. That could be done by means of War Savings Certificates on better terms than any office, ordinary or industrial, could give.

He believed that group insurance would, in all probability, lead to a very considerable increase in the amount of assurance taken of every kind. If the idea became at all common—and he certainly hoped it might—that no one should carry less than a year's salary as insurance, one would hope that a great many men who had industrial policies for £10 would regard such an amount as totally inadequate, and would want to increase their insurances to £50 or £100. However widely group insurance was extended, there would inevitably be an enormous number of individuals to whom no system of group insurance would be applicable.

Sir ALFRED WATSON said that the paper dealt with one of those superficially attractive schemes of insurance which for many years past they had been accustomed to receive from their cousins across the Atlantic, and which prudence dictated to them should be examined with very great care before they were engrafted upon our system of insurance in this country. He thought possibly that if the scheme were put before the employers here it would meet with a considerably less measure of acceptance than it appeared to have done in the United States, because employers in this country were already under considerable obligations in the matter of insurance, which they could not escape, and for which, as far as he knew, there was no counterpart in America. For example, the employers of 15,000,000 of people here had to pay every week for health insurance, and if one read the signs of the times aright, as gathered from statements at public meetings, and in the press, there was at least some probability that the liability of employers under that head was likely to increase in the near future. Again, four or five millions of people were insured against unemployment, and there again the employer was under the liability of making compulsory contributions; if they were to judge by statements that had been made, more or less officially, that scheme was more likely to be extended than the reverse, especially now that the out-of-work donations had familiarized a large part of the population with the meaning of unemployment benefits. It might, therefore, be that group insurance, if put before employers generally in this country,

would meet with a good deal less appreciation than it seemed to have met with elsewhere. His own feeling was that group insurance only needed examination to be severely criticized. The author made it perfectly clear that such insurance was contingent upon employment, although in some cases arrangements were made whereby people who left employment might continue their insurance at the rates of premium appropriate to their attained age. Let them consider that point in relation to the ordinary working of a superannuation fund. If they looked at the valuation reports they would generally find that the tables exhibited very low mortality rates during active service, a very heavy rate of mortality amongst those who had retired invalided, and a very heavy rate of retirement as the ages between fifty and sixty were reached. That meant that the risk of people falling out of their employment as they advanced in years, on account of inability to stand the constant strain of their work, was very considerable, and that a scheme of life insurance which only covered the active part of a man's life would fall very far short of the real insurance which the working population would think that they were obtaining from the arrangements made by the employer. Looking, again, at the experience of any typical friendly society, they would find that, taking the whole mass of the insured, something like one in seventy-two were in receipt of permanent sickness benefit. In the very nature of the case the great majority of those people must have been struck off the pay-rolls of the firms in whose service they were engaged before their incapacity occurred. A scheme of insurance which said, in effect, that so long as a person was in good health and able to follow his occupation, he was insured against death, but that as soon as the contingency of serious ill-health fell upon him and compelled him to retire from his active occupation, then the insurance ceased, would not, he thought, appeal to the high traditions of life assurance as practised in this country. Take, again, the case of the tuberculous—those unfortunate people who, at the age of thirty or thirty-five, were compelled to relinquish their employment and to go into a sanatorium. As a rule they began their sanatorium treatment some time after they had been compelled to relinquish their employment: the treatment itself extended over a considerable period, and in the great majority of cases he did not suppose that they would be kept upon the employer's pay-roll. He was afraid that in those cases, just when the insurance was most needed, it would fail; and he doubted whether a form of insurance having that result was one which they should encourage. He could imagine the dissatisfaction of the working classes on learning that after years of service in a particular business, followed by the misfortune of ill-health, the insurance had ceased, and that just when it was most needed there was no life assurance protection. He did not overlook the suggestion that a person who left an employment might, under some of the schemes adopted in America, provide for the continuance of his insurance by payment of the ordinary premium appropriate

to his attained age and without any further medical examination. Upon that he would like to say two things. First, as regarded the individual, there was the fact that the great risk of a premature breakdown in health occurred after the age of fifty, when the provision of a policy would be an extremely costly business, especially for a person whose income had ceased by reason of his incapacity. The fact that that option was very little exercised was exactly what one would have anticipated in the circumstances, and really enforced the argument that the option was no real protection. Secondly, in regard to the office which undertook the business, if the option was to be taken advantage of it must represent a very real liability, and it seemed to him almost superfluous to consider what was the appropriate premium on an orthodox mortality basis unless they could in some more or less accurate way gauge the value of the option and protect themselves against its universal exercise. The paper was certainly an interesting one in that it gave them a good deal of food for thought; but he hoped that, on reflection, it would lead them to think that in this country they were not prepared to adopt the particular form of insurance which it so clearly explained.

Mr. C. W. KENCHINGTON felt very considerable doubt whether the adoption of group insurance would benefit industrial policyholders. No doubt insurances under the group system would tend to diminish the expense ratio, but it was not clear to him that there would be any special advantage that would accrue to policyholders whose premiums continued to be paid weekly or at short intervals. The disadvantage of the system was that it did not give permanent insurance, and, although the industrial system might be criticized because in some cases the assured did not find himself in a position to continue the premiums, yet in those cases the lapse was at the free will of the assured and it did not depend upon his falling into ill-health—in fact, they knew that industrial policies were kept up during ill-health. Mention had been made of the option to continue the insurance when a man ceased employment. In that connection he was very much struck by a remark in an American insurance publication, in which the statement was made that it did not matter to any great extent whether the option was included in the policy or not, because even if the option was there it was very unlikely to be exercised.

Mr. ERIC B. NATHAN said that in regard to superannuation funds it was generally suggested that even if the employer paid the contribution, the money was really provided by the employees, being of the nature of deferred pay. If that view applied to group insurance they could see the difficulties that might arise if an employer ceased to insure or if the employee left his situation. Such a state of things would, he thought, be regarded by the working classes as a great hardship. Immediately the insurance was effected it would be looked upon as a right, and it would be very difficult to discontinue it; and it would seem that difficulties of this sort had already arisen in some parts of the United States. He was

rather surprised that Mr. McCormack had not referred to a form of insurance which had been in force in this country for a considerable number of years—certainly before the first group policy was issued in the United States; he referred to a system which obtained in the Co-operative movement, by which on the death of a member his representatives received the amount of his purchases during the previous year.

Mr. H. H. AUSTIN deprecated some of the author's remarks with reference to industrial insurance. Great numbers of the working classes did make substantial provision against death by means of ordinary and industrial branch policies. Every facility was given them to make that provision in the ordinary branch by the issue of policies as low as £25; premiums were accepted monthly as well as quarterly and half-yearly, and those facilities were taken advantage of very freely. Although the average industrial policy was only about £10, it was very usual to have two or more policies existing on the same life.

Mr. S. G. WARNER, in closing the discussion, said that Mr. McCormack had brought the subject of the paper before them in a way which, for clearness, comprehensiveness and good arrangement, left little room for criticism. He had been happily inspired in his choice of a theme, having regard to the present national position. They were entering on a period in which the relations between Capital and Labour had to be seriously re-considered, and the state of mind thus induced was one suitable for the consideration of experiments in social insurance. Such developments as group insurance could be regarded from two standpoints: as affecting national interests, and as offering attractive paths for private enterprise. The two things might quite possibly be combined, and they might fairly regard the business of life assurance as successfully effecting that combination. The time had come when private enterprise, which did not make adequate contribution to national benefit, would, by that fact, be condemned. By the American companies, which had taken up group insurance, the enterprise was held to be attractive and profitable. Naturally it had been compared with industrial insurance. Industrial insurance was expensive, and could not be otherwise, developed as it had been through the channels of individual appeal and individual collection. Obviously much cost could be saved were it possible to substitute the group for the unit, and place the burden of premium payment upon the employer. Group insurance, in effect, proposed to approach the employer and to insure his employees as a whole and in a group, without medical examination; the substituted safeguards being continuous employment for a specified number of months before assurance and compulsory inclusion in the group of all the employees. This made the group a miniature sample of the whole average population, so that the companies could take the risks one with another and assume that they were getting, on the whole, average lives. Another important point made was that the employer must bear the entire cost. There was, as Mr. Nathan had pointed out,

a danger that this might be considered by the workers an advantage more apparent than real, and that the cost might ultimately fall on the employee. To avoid this, it would be essential that wages and all existing benefits should remain entirely unaffected, so that there could be no possible doubt about the new benefit being an added privilege. Another feature of interest about which not much had been said in the discussion was the grading of rates according to employment hazard. In this respect the scheme, while it touched industrial insurance on one side, might be said to touch workmen's compensation business on the other; and, should it come into general operation, very valuable help would doubtless be found in the experience accumulated by the companies transacting workmen's compensation business. He thought it would be very desirable, if possible, that some common scale of rates for employment risks, based on the experience of the companies, should be agreed upon.

Coming to the important question of how the scheme was likely to suit industrial conditions in this country, it had been pointed out that it would conduce to continuity of service, but that the more it approximated to universal adoption the less this consideration would apply. It was open to question, however, how far such continuity would be an advantage. An able worker might not desire it. Probably the greater advantage would be that the scheme would tend to promote that more friendly and human relation between master and man which, could it be fully attained, would be the true solution of industrial difficulties.

One very important point arose as to what happened when the employment came to an end. The insurance offered was a temporary one, at rates fixed from year to year to cover the risk of each year, the group premium being revised annually on a statistical schedule; thus again linking the methods of the scheme to those of employers' liability business. At its termination the worker was offered a whole-life or endowment assurance of like amount, without medical examination, at the rate for the age then attained. Since this included all who left on grounds of health, it raised a question of some gravity. It has been urged that few of those thus entitled would be able to afford to avail themselves of the option, but that was hardly an argument which it would be worthy or safe to employ. Nor could they base any conclusions upon what might hitherto have happened in this respect, for the whole system was only about seven years old. As time went on what had happened with workmen's compensation would happen here: the workers would become more and more alive to the advantages offered them, and would more freely and generally take them. The origin of the option was the fact that some of the American States would not, without its inclusion, allow the business to be done within their borders. It seemed clear that if the option were offered here it must be paid for, or serious consequences might follow.

One thing more he would say, and it was suggested by the very pregnant and illuminating remarks of Sir Alfred Watson. This scheme came from across the Atlantic and sprang, therefore, from

a national history and national conditions very different from their own. The United States remained, at present, as they had been substantially throughout their history, the home of individualism. Little was done there for the worker by the State. He was highly paid, and left very much to fend for himself. Employment was abundant, and transition from one employer to another easy. In just such circumstances individualism flourished. Under that *régime* great fortunes had been made and a brilliant commercial fabric had been built up. The time would come, as population pressed more on the limits of employment, when some of the difficulties too familiar here might make themselves felt there also; and already, in fact, there were indications of the change. Here, as they all know, the State had already assumed and was ever more largely assuming a share in looking after the well-being of its people, and that by the pressure of compulsory legislation upon employers of labour. Often unwelcome, regarded with trepidation, entered on with hesitation, it inevitably came and grew. In a country which as yet knew little of these things, it was easy to appeal to employers to take upon themselves new burdens in the interest of the employed, and the movement was beneficent. Some large American companies made it their business to bring before large employers to whom they had granted group insurances, developments of public enterprise in the direction of caring for the health of the people, in sanitation, medical attendance and otherwise. It was done primarily from a commercial motive, but its effect could not fail to be a national benefit. How far the employers of labour in this country, burdened as they already were and were increasingly likely to be, with national responsibilities imposed by the State, might be expected to assume voluntary burdens it was difficult to say. It was to be hoped that if any practical development took place it would be in a form at once attractive to individual enterprise and defensible because a benefit to the people as a whole.

The PRESIDENT, in moving a vote of thanks to Mr. McCormack, said they were greatly indebted to him for his paper and for the interesting discussion which it had brought forth. With reference to Mr. Austin's criticism of the author's observations on industrial offices, he (the President) was partly responsible for those observations, as he mentioned the subject in his Address. He did not believe that Mr. McCormack regarded group insurance as a substitute for industrial assurance, nor certainly did he (the President), either when he first referred to the question or now. He did not regard it as anything more than supplementary to it. He thought, however, that possibly some of the methods which were employed in connection with group insurance might be of value in lessening the expense of industrial assurance. At present that expense was inevitable, but it was going to be a strong point of attack by those who were out to do something which, whatever form it might ultimately take, would not be of benefit to industrial assurance companies as they at present existed.

Mr. McCORMACK, in replying to criticisms, said that under

most group insurance schemes employees remained covered during any temporary absence on leave, and sometimes the employer had the option of continuing the insurance in the case of employees who retired or left through ill-health. But this privilege might be conditional upon the continued payment by the employer of at least a part of the salary. This was quite distinct from the option of conversion to an ordinary policy on leaving service; but in spite of these arrangements, he agreed with Sir Alfred Watson and other speakers that serious difficulties remained. It was rather a paradoxical feature of group insurance that not only its advantages but also some of its principal disadvantages were largely the result of the term insurance system, which was the basis usually adopted in practice. He thought the chief difficulty of the scheme from the point of view of the insurance company was to be found in the continued insurance of members who were no longer actively engaged. It therefore seemed very important that when the scale of premiums was fixed careful allowance should be made for the option of conversion to an ordinary policy on leaving service, and it would also be advisable to establish a reserve fund to provide for any losses which might result. With regard to Mr. Schooling's interesting suggestion a difficulty that occurred to him was that in any scheme involving group insurance it was necessary that all the employees in the group should be insured. Therefore the employer would have to insure employees who declined to take part in the scheme on the same footing as those who became members. But as a decreasing short term insurance, such as was suggested in the scheme, would probably be of little value apart from the scheme, the cost of insurance of non-members might represent a practically useless expense to the firm. He had not expressed any definite opinion of group insurance in the paper; but he thought he might safely say that so long as group insurance was regarded merely as an extension of Workmen's Compensation insurance, it might serve a valuable purpose.

The American-Canadian Mortality Investigation, 1900–1915.*

[Communicated by Mr. Arthur Hunter, A.I.A., F.F.A., Chairman of the Committee on Mortality Investigation.]

Synopsis of Vol. II.†

In Volume II of the report on the investigation into the mortality among residents of the United States and of Canada sub-divisions have been made of the statistics according to:

(*a*) Plan of insurance;

(*b*) Residence in groups of states in the United States;

* Published by the Actuarial Society of America.

† See p. 259 for synopsis of Vol. I.

(*c*) Residence in provinces, or groups of provinces in Canada ;

(*d*) Causes of death ;

(*e*) Amount of insurance $50,000 or more.

A synopsis of the results of the investigations is now given.

Mortality according to Plan of Insurance.

The mortality according to plan of insurance was confined to men resident in the United States at date of application for insurance, wartime conditions making it not advisable to undertake a similar investigation for men resident in Canada. The former are designated "American Men." The principal plans of insurance investigated were :

1. Ordinary Life, together with Endowment Insurances maturing at age 80 or over.

2. Twenty Payment Life, together with Nineteen Payment Life, and also Nineteen and Twenty Payment Endowment Insurances maturing at age 80 or over.

3. Twenty Year Endowment Insurances, and also Nineteen Year Endowment Insurances, both maturing at ages under 80.

4. Ten Year Term Policies, and Policies changed from Ten Year Term to other plans.

The first three groups largely consisted of policies on the Ordinary Life, Twenty Payment Life and Twenty Year Endowment Insurance plans.

In determining the expected deaths, the graduated rates of mortality for American Men (A.M.), both select and ultimate, were used. (See *J.I.A.*, li, p. 259). The following is a synopsis of the results :

Ordinary Life.

Duration	All Ages at Entry Combined		
	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
1–5...	$20,505,400	$19,062,200	107·6%
6 and sub.... ...	288,766,300	283,546,900	101·8
Total	$309,271,700	$302,609,100	102·2%

19 *and* 20 *Payment Life Plans.*

Duration	All Ages at Entry Combined		
	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
1–5...	$17,310,100	$18,817,000	92·0%
6 and sub.... ...	74,081,000	78,080,500	94·9
Total	$91,391,100	$96,897,500	94·3%

19 *and* 20 *year Endowment Insurances.*

Duration	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
1–5...	$5,279,300	$5,312,500	99·4%
6 and sub.... ...	25,449,000	26.878,600	94·7
Total	$30,728,300	32,191,100	95·5%

There is apparently an anomaly in that while the expected deaths for the first five insurance years are 7 per-cent of those for the sixth and subsequent insurance years for Ordinary Life policies, the corresponding percentage for Nineteen and Twenty Payment Life is 24 per-cent. This is due to the fact that the Companies contributed their entire data on plans other than deferred dividend for policies issued prior to 1900, while they contributed only a part of their data for issues of 1900 to 1914, inclusive.

The mortality on the Nineteen and Twenty Payment Life policies is better than on the Ordinary Life plan, and while the mortality on the former is slightly lower than on the Nineteen and Twenty Year Endowment Insurances, this may be due to accidental fluctuations. The experience in the United States with regard to the relative mortality under Limited Payment and Endowment Insurances is approximately the same as shown by the British Offices' Life Tables, 1893. In the case of the American companies, one reason advanced for the difference between these plans is that there has been a tendency to limit slightly impaired or border line risks to Endowment Insurances.

The difference between the relative mortality under the foregoing three principal plans was obtained by a method which took account of the distribution of business at the various ages and in the different insurance years. From this it appears that

the relative mortality is $12\frac{1}{2}$ per-cent lower on the Limited Payment than on the Ordinary Life plan, the difference being greater in the earlier than in the later insurance years; and that the relative mortality on Endowment Insurance is $7\frac{1}{2}$ per-cent higher than on Limited Payment Life in the first five insurance years, and practically the same for the sixth and succeeding insurance years.

An analysis is given in the report of the relative mortality by ages at entry for the first five insurance years, and by attained ages for the sixth and succeeding policy years. It is shown that under Ordinary Life policies the percentage of actual to expected deaths calculated by the A.M. Table (Select and Ultimate) was highest in the early policy years for ages at entry 35 to 44, and that under Nineteen and Twenty Payment Life policies the relative mortality was generally better in the earlier than in the later insurance years.

Term Insurance.—The mortality under term insurance was divided into two groups. The first was Ten Year Non-Renewable, or Ten Year Renewable Term Insurance exposed during the period of the Term Insurance, but not exceeding ten years. By limiting the period to ten years any selection against the companies at the beginning of the second period under renewable term policies was eliminated. The results were as follows:

Duration	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
1–5...	\$2,873,200	\$2,878,800	99·8%
6 and sub.... ...	1,878,700	1,769,900	106·1
Total	\$4,751,900	\$4,648,700	102·2%

For all ages and durations combined the relative mortality happens to be exactly the same as under Ordinary Life policies. This may cause some surprise, because a number of years ago the mortality of several companies was reported to be higher under Term Insurance than under Ordinary Life policies. In recent years, however, many of the companies have exercised a stricter medical selection, and made a more searching inspection on Term than on other forms of insurance.

The second group covered Ten Year Renewable Term policies renewed at the end of the ten years for a further period.

and those converted at that time to Ordinary Life, Limited Payment, or Endowment insurance—all without medical examination at date of renewal or conversion. As the experience begins at the end of ten years, the expected deaths were calculated by the ultimate table of mortality.

	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
All Insurance Years	$2,036,600	$1,878,600	108·4%

The mortality is divided according to number of years elapsed after renewal or change. For the first five insurance years following renewal or change the mortality is $9\frac{1}{2}$ per-cent higher than for the sixth to the tenth years after such renewal or change.

A division of the data into attained ages (*a*) under 60 and (*b*) over 60, showed 100 per-cent of the A.M.[5] Table under the former and 131 per-cent under the latter. This was to be expected, because the older the life the higher the cost of renewal or change, and accordingly the greater the incentive to continue the policy on an impaired risk.

Term Policies Changed to Other Plans.—An investigation was made of the experience under Ten Year Non-Renewable, or Renewable Term policies which were converted or changed into other forms before the end of the original term period without a new medical examination. As the mortality was likely to be influenced by the number of years elapsed since the original policy was issued, the experience was divided according to the number of years elapsed before change or conversion was made. The expected deaths were calculated by the A.M. Table, making allowance for the time elapsed before change. The following is a synopsis of the results :

	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
Changed during first five calendar years...	$482,100	$573,500	81·5%
Changed during sixth and subsequent calendar years... ...	310,500	319,200	97·3
Total	$792,600	$892,700	88·8%

As the policies were changed to a higher premium plan it was not to be expected that there would be an unfavourable mortality in this class, except, possibly, for those changed in the later years of the Ten Year Non-Renewable Term period. In the case of companies which permitted the conversion within one year of the end of the Non-Renewable Period, a higher mortality than the normal for a few years following the change might be anticipated, but the statistics were not numerous enough to justify such a sub-division.

Mortality in Different Sections of the United States.

The data for the United States were divided into ten groups, depending upon the State in which the man resided at the date of application for insurance. These groups took account of the geographical situations and economic conditions of the States. The ten groups were as follows:

1. Connecticut, Maine, Massachusetts, New Hampshire, Rhode Island, Vermont.
2. New Jersey, New York, Pennsylvania.
3. Delaware, District of Columbia, Kentucky, Maryland, North Carolina, Tennessee, Virginia, West Virginia.
4. Illinois, Indiana, Michigan, Missouri, Ohio, Wisconsin.
5. Florida, Georgia, South Carolina.
6. Alabama, Arkansas, Louisiana, Mississippi.
7. Iowa, Kansas, Minnesota, Nebraska, North Dakota South Dakota.
8. Oklahoma, Texas.
9. Arizona, Colorado, Idaho, Montana, Nevada, New Mexico, Utah, Wyoming.
10. Alaska, California, Oregon, Washington.

The relative mortality in these States for all insurance years and for all ages at entry combined may be seen from the following synopsis, the expected deaths being calculated by the A.M. Table:

Group	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
1	59,486,800	62,260,700	95·5%
2	150,749,200	146,499,500	102·9
3	43,133,300	41,815,100	103·2
4	108,017,800	114,032,900	94·7
5	13,703,600	11,073,600	123·8
6	13,181,500	10,626,400	124 0
7	21,533,200	24,289,300	88·7
8	7,143,200	6,557,400	108·9
9	5,952,900	6,005,900	99·1
10	13,241,500	13,185,600	100·4

The lowest relative mortality was in the middle west (7) in the States where the principal occupation is farming. There was very little difference in the mortality between the New England States (1) and the group of Northern Central States (4). The Eastern States, extending from New York to Virginia (2 and 3) showed a slightly higher mortality than the Rocky Mountain States (9), and the Pacific Coast States (10). As was expected, the two groups (5 and 6) of extreme southern States showed the highest mortality—24 per-cent in excess of the average for the whole country.

It is interesting to note that the older settled States and States with large cities, had a lower relative mortality in the earlier than in the later policy years, while the reverse was true in the other States.

Mortality by Provinces in Canada.

An investigation was made by groups of Provinces of the mortality among Canadian men, according to their residence at date of application for insurance. The following divisions were made :

(A) Ontario.
(B) Quebec.
(C) Nova Scotia, New Brunswick and Prince Edward Island.
(D) Manitoba, Alberta and Saskatchewan ; Yukon Territory.
(E) British Columbia.

The following shows the comparison of the actual with the expected deaths calculated by the C.M. Table :

Group	Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
A	14,435,300	15,284,200	94·4%
B	7,216,800	6,098,800	118·3
C	3,856,900	3,760,200	102·6
D	2,670,700	3,063,600	87·2
E	1,207,200	1,174,900	102·7

The best mortality was found in the wheat-growing States of Western Canada (D). It will be remembered that the best mortality in the United States was in the farming communities of the Middle West. The mortality in the Maritime Provinces (C) and in British Columbia (E) were practically alike, $2\frac{1}{2}$ per-cent higher than the average for the whole of Canada. The Province

of Ontario (A) was $5\frac{1}{2}$ per-cent below the average. The highest mortality was in the Province of Quebec (B), which was about 30 per-cent higher than for the Western Provinces (D). This is generally ascribed to the higher mortality among the French-Canadian population of the Province of Quebec.

An investigation was made by request of the Canadian actuaries of the mortality among women in the Province of Quebec. The amount of data was small, the actual death losses being $270,700, and the expected $241,600—a ratio of 112 per-cent. As the mortality among Canadian women was 10 per-cent lower than among Canadian men, it follows that the mortality among Quebec women was 25 per-cent higher than among all Canadian women.

Causes of Death.

An investigation was made of the causes of death in the case of both American and Canadian men. The deaths were tabulated in two ways: (*a*) percentage of deaths from each cause to total deaths; (*b*) death-rate from each cause per $10,000 exposed to risk. Nothing unexpected seems to have been developed by these tables. On the basis of the rate per $10,000 exposed to risk an increasing death-rate with advance in age is exhibited from such diseases as cancer, pneumonia, apoplexy, heart disease and nephritis. The death-rate from typhoid fever decreased until after middle life. It is only possible to make a general comparison of death-rate by causes with those in the Medico-Actuarial Mortality Investigation, and from this it appears that there has been a material reduction in the death-rate from typhoid fever for all policy years, and a marked improvement in the first year's experience in the case of tuberculosis of the lungs.

A comparison of the causes of death among American and Canadian men indicates that there was a markedly higher death-rate among the former than among the latter, from apoplexy, heart disease and nephritis, but in the case of accident the Canadian death rates were the higher.

Policies for Large Amounts.

Two investigations were made into the mortality under policies for large amounts: (*a*) policies for $50,000 or more, and (*b*) insurances of $100,000 or more, taken out in amounts of at least $50,000 per policy. This investigation was limited to the data on American men. Unfortunately, the amount of data is

small, because a small part only of the statistics of the large companies was included; the larger the company, the smaller was the proportion of data contributed for the years of issue 1900 to 1914 inclusive. The small and young companies contributed all or a large part of their experience, but there were naturally few policies issued by them for amounts of $50,000 and more.

The expected deaths in the following synopsis were calculated by the A.M. Table:

Policies for $50,000 *or more.*

Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
$6,423,700	$5,502,500	116·7%

The relative mortality for the first five policy years combined was the same as for the sixth and succeeding policy years combined.

On policies with aggregate amounts of $100,000 or more on individual lives the following was the result:

Aggregate Amounts of $100,000 *and over.*

Actual Deaths	Expected Deaths	Ratio of Actual to Expected Deaths
$2,595,500	$1,978,100	131·2%

The mortality in the first five insurance years combined was lower than in the sixth and succeeding insurance years combined; but too much reliance should not be placed on this result, because of the small amount of material.

From the two foregoing synopses it may be deduced that there was a ratio of actual to expected deaths of 109 per-cent on cases for at least $50,000, but less than $100,000.

Information was obtained from three companies with regard to their experience on large amounts of insurance. The actuary of one company stated that, under policies issued from 1880 to 1915 for an aggregate amount on individual lives of $100,000 or more, issued within a single calendar year, the mortality was about 25 per-cent in excess of the normal mortality of the company on all its insurances. The amount of data in this company's experience was about three times as large as

those investigated in connection with the "American Men" experience.

Another company gave its experience on policies issued for $50,000 or more from the years 1868 to 1907 inclusive, carried to the anniversaries in 1908. The select table adopted in this experience in determining the expected deaths was probably higher than the general experience of the company, and, accordingly, the ratios now given are too low. On persons who were insured for at least $50,000 but less than $100,000 the relative mortality was 107·5 per-cent, and on those insured for $100,000 and over it was 113·2 per-cent. These ratios are on the basis of "lives"; the corresponding percentages by "amounts" are not materially different.

A third company gave its experience under policies issued in the years 1885 to 1915 inclusive, carried to the policy anniversary in 1916. On policies issued for $50,000 to $100,000 inclusive the relative mortality measured by the Medico-Actuarial Mortality Table was 107 per-cent by applications, and 105 per-cent by amounts. For cases about $100,000 the relative mortality was 110 per-cent by applications, and 148 per-cent by amounts insured.

A recent tendency has been to greatly increase inheritance taxes, and, accordingly, the companies are receiving far more applications than formerly from persons of wealth, who wish to provide for these taxes by means of insurance. Furthermore, it is becoming the practice of companies to insure their principal officers and technical men for the benefit of the company. It is likely, therefore, that there is less "moral hazard" than formerly in connection with applications for large amounts.

It is not the intention to make any further investigations, and the work of the committee is completed, unless the A.M. or the C.M. Tables are likely to become the standards of valuation in the United States or Canada, in which event complete sets of terminal and mean reserves will be prepared and published.

While no additional mortality studies will be made by the committee in charge of the American-Canadian Mortality Investigation, three investigations have been commenced by a Joint Committee of the Actuarial Society and the Medical Directors' Association. That joint committee has already published a report on the mortality due to overweight, and is now engaged in studies of the effect of family history on longevity and the mortality from functional heart murmur and intermittent albuminuria.

LEGAL NOTES.

By WILLIAM CHARLES SHARMAN, F.I.A., *Barrister-at-Law.*

Voluntary assignment of a *chose in action.*

ALTHOUGH before the Judicature Act, 1873, the rule was that equity would only grant its aid to compel an assignor to allow the assignee of a legal *chose in action* to sue at law in his name when valuable consideration has been given for the assignment, section 25, sub-section 6, of that Act has not only improved the position of the assignee as regards procedure by enabling him to sue in his own name, but has also enabled him so to sue in regard to a debt which has been assigned voluntarily.

In the case of *In re Westerton, Public Trustee* v. *Gray*, reported 88 L.J. Ch. 392 (1919), 2 Ch. 104, the validity of a voluntary assignment of a *chose in action* was considered. The facts are as follows :

A testator about a year before his death handed to his landlady an envelope enclosing three documents, saying : " Here is " a present for you ; that is for you. I have given it into your " hand." She was about to open it, when he took it back from her hand, and said : " Remember, I will keep it for you," and then placed it in his despatch box. The envelope contained (*a*) a letter under the hand of the testator and addressed to the landlady as follows : " You have been very kind to me, and " I desire to make some return by giving you the amount of " £500, now on deposit at the . . . . . Bank as per receipt " enclosed " ; (*b*) a banker's deposit receipt for £500, which he had in 1914 placed on deposit at his bank in his name ; (*c*) an authority under the hand of the testator to the bank to pay the landlady or her order the sum of £500, then on deposit there. The deposit note was not transferable, and the endorsed form of receipt was not signed by the testator, and no notice of assignment was given to the bank until after his death, in May 1917, when the envelope was taken out of the despatch box and found to contain these three documents.

Sargant, J., held that the letter was a direct assignment to the landlady of the money on deposit ; that upon the delivery of the documents to her there was a completed gift, the subsequent redelivery to the testator being merely for their safe custody on her behalf ; and that by virtue of section 25, subsection 6, of the Judicature Act, 1873, she could sue for the money

on deposit, although it had been assigned to her voluntarily, and she was therefore entitled to have it paid to her.

The facts in this case should be carefully distinguished from those in the case of *In re Williams, Williams* v. *Ball*, previously reported in these notes, *J.I.A.*, vol. l, pp. 118 and 243.

Policy on life of bankrupt concealed from trustee. Conflict of claims to policy moneys.

The judgment of Horridge, J., in the case of *In re Stokes, Mellish ex parte*, reported 88 L.J.K.B. 794. deals with a point which frequently arises in office practice, and usefully reviews several previous decisions on the same point, some of which decisions have previously been reported in these notes.

In this case a motion was brought to obtain a decision of the Court as to whether a policy of assurance on the life of the deceased passed under his will to the applicant, or whether it belonged to the Official Receiver as trustee under the following circumstances. On 1 December 1879, the debtor filed his petition for liquidation, and on 13 December 1879, the creditors resolved that the debtor's affairs be liquidated by arrangement, and not in bankruptcy. In January 1880, a trustee was appointed under the liquidation who carried on the business of the debtor for practically the whole period of the liquidation and to enable him to do so employed the debtor as his clerk. On 21 October 1881, the debtor effected a policy on his life for £1,000 at an annual premium of £33 10*s*. and continued to pay the premiums until his death. On 7 November 1883 the liquidation was closed, and on 13 November 1883 the debtor obtained his discharge. On 26 July 1917 the debtor died at Cape Town, South Africa, having by his will, dated 10 July 1915, appointed his niece, the applicant, his executrix. Letters of administration, as proof of probate of the said will, were granted to the said executrix by the Master of the Court at Cape Town, and were forwarded, together with other documents, by the applicant's solicitors in London to the insurance company; but the company declined to pay the policy moneys to the solicitors on behalf of the applicant until it had been determined whether the same passed under the will of the deceased to the applicant or belonged to the trustee. The trustee stated on the hearing of this motion that he did not know the policy was in existence, and that he knew of no premiums being paid by the debtor, and that the premiums must have been paid out of the debtor's salary. If he had known of the policy he would have disclaimed it.

Horridge, J., said: "This is a case where the policy was "effected and the premiums paid from first to last without the "knowledge of the trustee or any successive officer representing "the trustee. In the case of *In re Tyler, Official Receiver ex parte* "(1907) 1 K.B. 865 [*J.I.A.*, vol. xli, p. 411] the Court of Appeal "affirmed a decision by Bigham, J., in which he declined to "allow the trustee to intervene and take policy moneys without "making an allowance to the wife of the debtor who had paid "all the premiums; but in that case the whole matter had been "done with the cognizance of the representative of the estate "for the time being, although it is true he was not the main "representative involved in the case itself. That case entirely "proceeded upon the principle that the trustee, having stood "by with knowledge and allowed the premiums to be paid by "the wife, could not intervene and take the benefit of the policy "as an honest and upright man without allowing the wife the "amount of the premiums paid by her. It is put quite tersely "in the judgment of Farwell, L.J., when he deals with the case "of the wife as follows: 'He knew the wife was paying, and he let "'her go on paying, and said nothing to her to lead her to "'believe that he was going to claim the policy moneys at the "'end without repaying the premiums which she had paid.' "That was the principle of *In re Tyler*. It appears also in the "case of *In re Hall, Official Receiver ex parte* (1907) 1 K.B. 875 "(*J.I.A.*, vol. xli, p. 413) in the judgment of Farwell, L.J. "He says: 'In the last case (*In re Tyler*) for example, my "'judgment proceeded on the knowledge of the trustee in the "'bankruptcy of the existence of the policies, and the necessity "'for paying the premiums, and the fact that the wife was "'paying them.' The next case is the case of *Tapster* v. *Ward* "(1909) 101 L.T. 25, 503 (*J.I.A.*, vol. xliv, p. 87), a decision of "Eve, J., and the Court of Appeal. Let me compare the facts "of that case with the facts of the present case. In that case "the policy was effected before the liquidation. The policy was "concealed from the trustee, and after the liquidation had been "closed, premiums were paid out of his own moneys by the "liquidating debtor from the year 1880 to the year 1907— "twenty-seven years. Now in this case it is quite true that the "policy was effected during the liquidation, and the first premium "was paid out of moneys which had been allowed to the debtor "by the trustee in respect of his salary. It does not make any "difference that the policy in Tapster's case had been effected

"before the liquidation, and that in this case the first premium "was paid on the policy at a time when the liquidation had "commenced. It is not disputed that every contract made "on and before the liquidation by the liquidating debtor is the "property of the trustee. If that is so, this contract of insurance, "although the liquidation was in fact proceeding, was the con- "tract of the trustee, and if the trustee had known about the "premiums being paid out of the debtor's salary the trustee "could have intervened, and said: 'I want this policy.' As "regards the subsequent payments which have been made "from the discharge in 1883 until the day that the assured died "in July 1917, they are exactly the same as the payments for "the policy in *Tapster* v. *Ward;* they were paid out of the "moneys of the liquidating debtor, and after he had got his "discharge. The Court of Appeal there refused to listen to the "argument that the trustee ought not to intervene and claim "the policy moneys. . . . . The last case upon the point is "a decision of my own, *In re Phillips* (1914) 83 L.J., K.B. 1364. "In that case the policy had been effected during the time that "the bankrupt's discharge was suspended under the Bankruptcy "Act, 1883, and subsequent premiums had been paid by him "until he became a bankrupt a second time. The question arose "whether the trustee in the first bankruptcy could claim the "policy moneys without accounting to the trustee in the second "bankruptcy for the premiums which had been paid by the "bankrupt in the interval. I held that, as the trustee in the "first bankruptcy had no notice whatever of the payment of "premiums, he was not bound in the least to make provision "out of the policy moneys. It seems clear law from the case "of *In re Leslie, Leslie* v. *French* (1883) 52 L.J., Ch. 762, that "a mere volunteer paying premiums has no right to treat them "in any way as salvage remuneration to him for keeping the "policy going. I think this rather a hard case for the lady, but "the law of bankruptcy seems to be prefectly clear. In this "case the policy was effected after the liquidation resolution, "and the policy belonged to the trustee. The bankrupt never "told him that he had effected it; he never told him that he "was paying any premiums. On the authorities I have referred "to I think it is clear that the policy must belong to the Official "Receiver. I declare the policy to be part of the estate and the "Official Receiver entitled to the policy moneys."

Relief from super-tax in respect of life premium paid under contract.

The decision of Mr. Justice Sankey in the case of *Earl Howe* v. *Commissioners of Inland Revenue* (1918) 2 K.B. 584, reported in these notes, vol. li, p. 135, has now been reversed by the Court of Appeal. The appeal is reported 88 L.J., K.B. 821—C.A.

The facts are as follows :

The Finance (1909–10) Act, 1910, section 66, sub-section 2, enacts that for purposes of super-tax the total income of an individual shall be estimated in the same manner as the total income is estimated for the purposes of exemption under the Income Tax Acts. The Income Tax Act, 1842, section 163, provides that a person chargeable to income tax shall be exempted if he shall prove, " according to the several rules and directions " of the Act, that his income is less than £150. Section 164 provides that a person claiming exemption under section 163 must deliver a declaration of " every sum of annual interest or other annual " payment reserved or charged " on his income " whereby the " income shall or may be diminished." Section 190, Schedule G, with the rules and directions therein contained, is, in making returns of annual value, to be observed by each person ; and Rule xvii, heading 3 thereof, provides that he must make a declaration of the amount of the interest, annuities or other annual payments to be made out of the property assessed. The Finance Act, 1916, section 36, sub-section 1, abolishes the abatements allowed in respect of premiums on life assurance policies for the purpose of super-tax. Earl Howe mortgaged his life interest in certain estates, and also assigned certain policies of assurance on his life by way of mortgage and further charge to an assurance company, covenanting to pay the interest on the sums advanced and the premiums on the policy. In case he should neglect to pay the premiums he empowered the company to pay them and to charge them upon the mortgaged premises. Earl Howe had always paid the interest on the sums advanced and the premiums on the policies, and claimed to be allowed to deduct these premiums, as being annual payments charged on his income from his income for the purposes of super-tax. The Commissioners refused to allow them to be deducted. Sankey, J., contrary to his own opinion, but following the decision of the majority of the Irish Court in *Massy* (*Lord*) v. *Inland Revenue Commissioners*, allowed them to be deducted. The Commissioners appealed.

Their Lordships allowed the appeal. They said that the

premiums were not "annual payments" within the Income Tax Act, 1842, sections 164 and 190, Schedule G, Rule xvii, as only annual payments from which income tax could be deducted by the payer were "annual payments" within these sections, and that the premiums could not therefore be deducted.

Are profits or gains on Treasury Bills subject to Income Tax?

A decision on the important question whether a Life Assurance Company, which has not been charged to income tax in respect of its profits or gains under Case 1 of Schedule D, should be subject to income tax in respect of the profits or gains arising from the sale or maturity of Treasury Bills under the third case of Schedule D, was given in the case of *National Provident Institution* v. *Brown*; *Provident Mutual Life Assurance Association* v. *Ogston*, T.L.R. 35, 690.

This case was an appeal from a decision of the Special Commissioners to the effect that such profits or gains were assessable to income tax, provided this source of income had not ceased before the year of assessment. Both the Institution and the Surveyor appealed against the decision of the Special Commissioners and the case was heard before Mr. Justice Rowlatt, who decided that such profits or gains were assessable to tax, whether or not there had been any similar transaction in the year of assessment. The facts of the case are as follows:

In each of the years ended 5 April 1916 and 5 April 1917 the Institution bought at the Bank of England Treasury Bills. Some of them were held until maturity. Others were sold in the open market during their currency, and the rest, being the whole of the Treasury Bills then held by the Institution, were converted early in 1917 into Five per-cent War Loan, 1929–47, on the terms of the prospectus issued on 11 January 1917. In the year ended 5 April 1918 the Institution did not hold or have any transactions in Treasury Bills.

In each of the years ended 5 April 1917 and 5 April 1918 the Institution received and paid interest, from which income tax was not deducted, on short loans to and from bankers. In the year ended 5 April 1918 the Institution received interest on Five per-cent War Loan, 1929–47, Stock and Bonds, income tax being deducted from the interest on the bonds, but not from the interest on the stock.

The total amount of the difference between the amounts paid and the amounts received by the Institution in respect of the bills held to maturity, and in respect of the other bills which were sold during their currency in the open market in the year ended 5 April 1916 was £5,422 11*s*. 7*d*. In the year ended 5 April 1917 the Institution bought Treasury Bills some of which they held to maturity and others of which they converted into War Loan. The total difference between the amounts paid by the Institution and received by them in that year was £19,714 8*s*. 6*d*. There was also an item of £851 19*s*. 6*d*. the difference between the interest paid and received on short loans to and from bankers. This made a total of £20,566 8*s*. for the year ended 5 April 1917.

The Institution was not, for any of the years 1916, 1917 and 1918 assessed to income tax under Case 1 of Schedule D on the balance of its profits and gains. For the year ended 5 April 1916 the only direct assessment made upon it under Schedule D of the Income Tax Acts was for income from foreign securities. For the year ended 5 April 1917 the first of the assessments under appeal was made upon the basis of the amount of the differences between the amounts paid and the amounts received on Treasury Bills realized within the preceding year. For the year ended 5 April 1918 the second of the assessments under appeal was made on the basis of the amount of the differences between the amounts paid and the amounts received on Treasury Bills realized, together with the amount of the difference between the interest paid to and the interest received from bankers on short loans within the preceding year. The Institution did not dispute its liability to assessment to income tax for the year ended 5 April 1918 for interest on short loans, on the basis of the net amount received within the preceding year, and the only questions raised by it related to its liability to assessment on the differences between the amounts paid and the amounts received by it for Treasury Bills.

It was contended on behalf of the Institution (*a*) that the differences between the amounts paid and the amounts received for Treasury Bills were an accretion of capital, and not income or annual profits and gains chargeable to income tax, (*b*) that the tax, if assessable at all, was assessable only at payment of the bills on maturity, and on the person holding them at that date, and that the bills sold or converted into War Loan during their currency should be left out of account in computing any liability on the part of the Institution, and (*c*) that in any case

the assessment for the year ended 5 April 1918 could not be maintained, as the Institution did not hold or have any transaction in Treasury Bills in that year.

It was contended on behalf of the Surveyor of Taxes, *inter alia*, (*a*) that the sums in question, whether received on maturity or on sale or conversion of the bills, were profits on discounts chargeable to income tax under the third case of Schedule D of the Income Tax Acts; (*b*) that a person was liable to income tax under the third case of Schedule D on the basis of the full amount of the profits or gains arising from the sources comprised in that case within the preceding year, whether any profits arose to him from such sources within the year of assessment or not; (*c*) that the Institution was in receipt of profits of a description comprised in the third case of Schedule D in each of the years of assessment and was consequently chargeable to income tax under that case in each of those years on the basis of the full amount of the profits or gains from any source comprised in that case within the preceding year.

Mr. Justice Rowlatt, in the course of his judgment, said: "It seems to me in each case that one must look at the real "nature of the transaction and see whether the purchase of the "future obligation at a discount is really an investment of money "at interest or not. Now, in the simple case of the purchase of "a Treasury Bill bearing no interest for such a sum as with "interest at such and such a rate for the currency of the bill "will give the face value, I can feel no real doubt that the "transaction is simply one of lending money at interest. If a "twelve months' bill for £105 is sold for £100, surely the purchaser "simply lends £100 for a year at 5 per-cent. If the face value "is £100, and the present value has to be reached by a sum in "proportion not giving a round sum in sovereigns as the result "the transaction is the same. Unfortunately the problem "involved in these appeals is not confined to this simple case. "In the first place, these companies have been assessed in the "second year, not merely on bills purchased and realized within "the preceding year, but on differences between the amount "received in that year and the purchase price given before that "year. On this principle a twelve months' bill bought in May "1914, and paid in May 1915, would be treated as yielding the "whole of the profits and gains represented by the discount in "the year ended April 1915. This cannot be right. This is a "practical matter, and cannot be treated as adjusted by com-

"pensation over a series of years. In the first place, the rate of "tax may change so that it is material to be exact in determining "in what year the income is taxable. Furthermore, these "Treasury Bills were sold to individuals as well as to corporate "bodies, and if one year is to be treated as bearing what is really "one and eleven-twelfths of a year's income, the difference for "supertax or abatement purposes may be of great importance. "I am not insensible of the difficulty of treating interest accrued "but not encashed as profits of the year when the question is "not of taxing a trading concern on the profits shown by its "balance sheet, but of taxing interest or discounts as such. "Still, the step must, as it seems to me, be taken. The difficulty "of adjustment as between the years does not lead me to modify "my view that the profit in itself is taxable as an annual profit "or gain.

"A more formidable complication arises where the Treasury "Bill is not bought from the Government or is not held to "maturity, but is either bought or sold in the market, or both. "I treat the so-called conversion of a Treasury Bill into War "Loan as being in substance as it was in form a sale to or a "discount with the Government at a price representing the "original price plus the proportion of interest or discount "accrued. But where a Treasury or any other bill is bought or "sold in the market the price depends on the market rate for "money. A person who has held the bill for, say, two months "may, in a period of pressure affecting himself and the market "generally, have to dispose of it at a price involving a loss of "the whole or part of, or more than, the interest for the time "during which he held it. Conversely, in times of ease, he may "sell it at a profit exceeding that interest, consequently, by the "time the bill matures, the bill may have passed through the "hands of half a dozen persons who have made profits out of "it aggregating a larger sum than the difference between the "issue price and the face value, such extra profits being, of "course, exactly equalled by losses made by other holders. "How is this situation to be dealt with? Is the difference "between the issue price and the face value to be treated as the "taxable income on the bill and to be all assessed on the holder "at maturity, or to be divided among the successive holders "proportionately to their periods of holding? In neither case "would the solution correspond with the truth. They have not "received such profits. In my judgment, the matter must be

"dealt with (subject to an adjustment between the years as "already indicated) in the way the Special Commissioners have "dealt with it—that is to say, the difference between the amount "paid on purchase and that received on realization must be "treated as a profit on a discount within the second rule of the "third case. If any holder has made a loss he drops out to the "extent of the loss, and the aggregate of the differences received "will bear tax to the Revenue, even though that aggregate "exceeds the difference between the original purchase or issue "price of the bill and its face value.

"This disposes of the appeals by the companies subject only "to two observations. Some of the bills in the second year in "the case of one of the companies were French Treasury Bills of "a currency, as I gather, of twelve months or less as in the case "of the British bills. Nothing was said in the course of the "argument to suggest that there was any distinction between "French and British Bills for this purpose, and I treat them, "therefore, as covered by my decision. The other point is this. "In the case of the same company and in the second year some "of the documents were not Treasury Bills, but War Expenditure "Certificates. The case gives me no information as to the nature "of these documents, nor do I remember to have been informed "of it at the Bar. I must assumed that they are on the same "footing for this purpose as Treasury Bills. The appeals by the "companies are therefore dismissed."

As regards the appeal of the Surveyor of Taxes against the Commissioners' decision relating to the assessment for the year ended 5 April 1918, the judge ruled that "in the case of profits "from discounts there is no existing source to be looked for in "the year of assessment, in order to support the tax, and the "appeal of the Crown must therefore be allowed."

ACTUARIAL NOTES.

Graphic Method of obtaining the Yield on a Redeemable Security. By O. F. Diver, M.A., F.I.A.

Let OX, OY be rectangular axes (Fig 1), and let AB be a graph of the function $a_{\overline{n|}}^{-1}$ for argument i, the latter being measured from O along OX. Then, if P be a point on the

graph corresponding to a particular value i of the argument, and PN be drawn perpendicular to the base-line OX,

$$PN = a_{\overline{n}|}^{-1} \text{ for the value } i = ON.$$

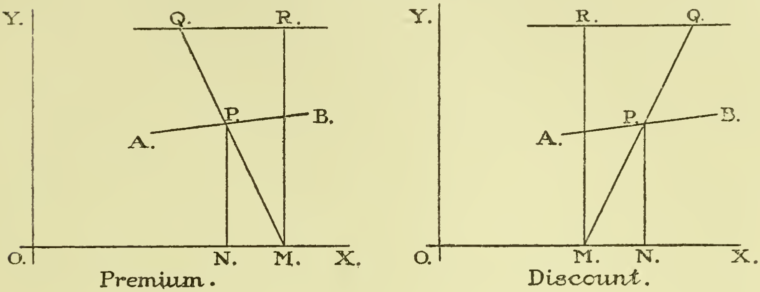

Fig 1.

On OX take another point M, where $OM = g$. Join MP, and produce it to meet at Q the line QR parallel to OX at a distance C from it. Draw MR parallel to OY meeting QR in R. Then MR = C.

By similar triangles MRQ and PNM,

$$QR : MR = MN : PN,$$

or

$$QR = C(g-i)a_{\overline{n}|}.$$

Now, if we have a security redeemable at C n years hence, and paying an annual dividend gC in the meantime, this equation shows that, when the security is valued at the rate i, QR is the premium over the redemption price C, or the discount below it, according as $g >$ or $< i$, or as Q lies to the left or right of R.

Inversely, if QR is the premium or discount, and MR the redemption price, the line MQ will cut AB at the point P, whose abscissa $ON = i$, the yield on the security. Thus, if we draw a series of graphs of $a_{\overline{n}|}^{-1}$ on squared paper for various values of n, we can obtain in this way the yield on any security redeemable at a fixed price in any given number of years, and paying any given rate of dividend. If $a_{\overline{n}|}^{(2)}$ is used instead of $a_{\overline{n}|}$ for the graphs, the method will be applicable to the almost universal case of half-yearly dividends, and the yield will be given in the form of a nominal annual rate convertible half-yearly.

The method also lends itself to the calculation of net rates of yield, after allowing for income tax, as it is equally easy to use with gross or with net rates of dividend.

There is more than one practical method of finding the point P without actually drawing the line QM on the diagram. As it will be found convenient in practice not to measure PN or MR on the same scale as ON, or as each other, let us suppose that on the scale on which ON is measured,

$$\text{PN} = l/a_{\overline{n}|}^{(2)} \text{ and } \text{MR} = m\text{C}$$

where l and m are constants.

Then, if k is the premium (or discount, if negative), the equation shows that $\text{QR} = \frac{m}{l}k$. *E.g.*, if $l = 0{\cdot}1$, and $m = 0{\cdot}01$, then $\text{QR} = k \div 10$.

Now, if we take first for simplicity the case of a security redeemable at par, QR will be a horizontal line drawn on the diagram at distance m from OX. If a scale of rates of interest be written in along this horizontal line, corresponding to that along the base-line OX, Q lies at the point $g - \frac{m}{l}k$ on this scale. A ruler or straight-edged card may now be placed, so as to pass through the point g on the base-line, and the point $g - \frac{m}{l}k$ on the horizontal line. It will then cut the graph corresponding to the outstanding term n, at the point whose abscissa is i. The yield may then be read off by carrying the eye to one of the interest scales.

If the security stands at a discount, k is negative, and $g - \frac{m}{l}k$ is greater than g, so that Q will lie to the right of R.

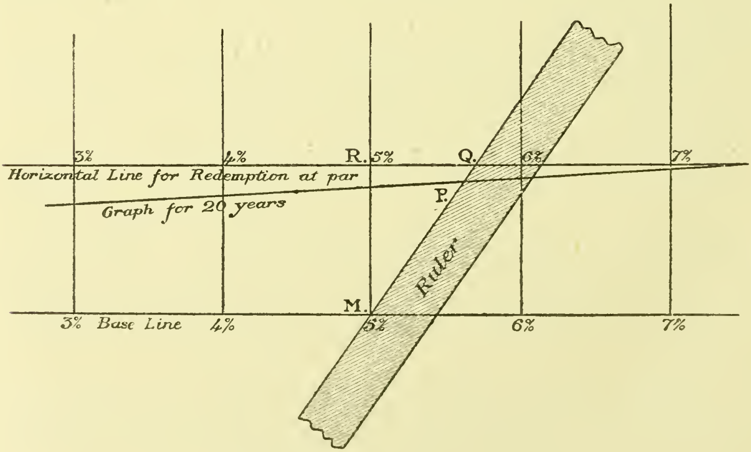

Fig 2.

Fig. 2 gives an example of the case of a 5 per-cent Bond, redeemable at par, and now standing at a discount of 7 per-cent, just after an interest date. In this figure the values taken above for l and m are used, so that the ruler has to pass through the point 5·7 per-cent on the horizontal line.

This is the most obvious method, and the modifications necessary in the case of a security not redeemable at par will readily occur to anyone sufficiently interested to try the method practically.

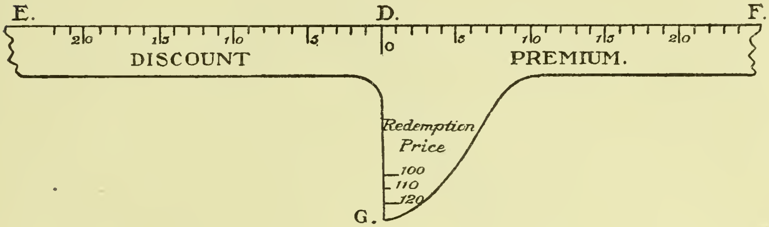

Fig 3.

The writer has found by practical experience that the following alternative method of placing the straight-edge is somewhat quicker, especially if the security is not redeemable at par. For this we require a specially shaped ruler (see Fig. 3), with a straight edge EDF, and a transverse arm DG, the edges EF and DG being at right-angles to one another. Along DE a scale of discount is marked off, along DF one of premiums, and along DG one of redemption prices, all scales being measured from zero at D, the middle point of the edge EF. In this method no horizontal line is required on the diagram, except the base line.

Fig. 4 shows how this method works in practice for the same example as before, namely, a 5 per-cent Bond redeemable at par, and now standing at a discount of 7 per-cent just after an interest date.

The ruler is placed on the diagram, so that the straight edge EF cuts the base line at 5 per-cent on the interest scale as before, but so that the base line also cuts the straight edge at the 7 on the discount scale. The ruler is manipulated till the base line also cuts the scale of redemption prices (DG) at 100 (T). The straight edge now cuts the graph at the required point.

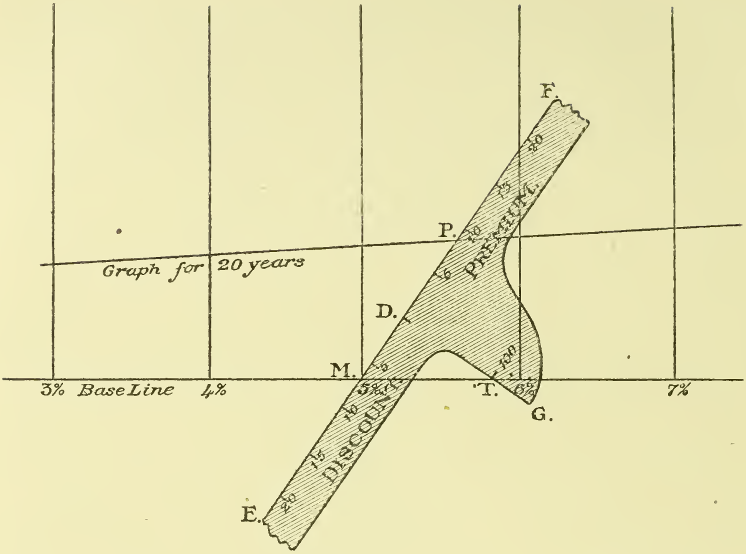

Fig. 4.

In the special case of a security not redeemable at par, it is only necessary to substitute for the nominal rate of dividend the equivalent rate calculated on the redemption price, and to adjust the ruler so that the base line cuts the scale of redemption prices at the actual price instead of 100.

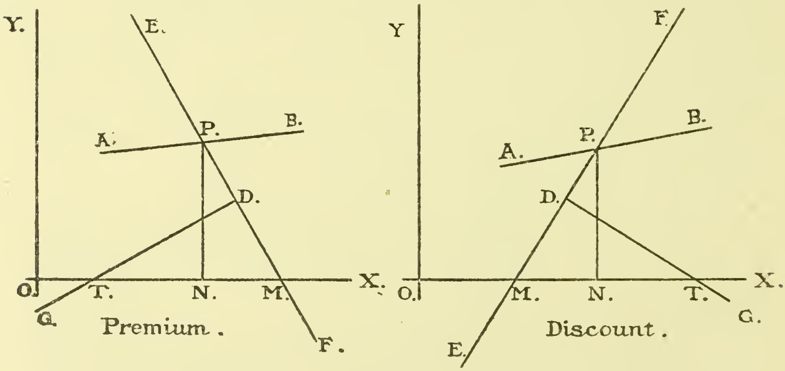

Fig. 5.

The theory of this method is as follows. In Fig. 5 the lines EDF and DG correspond to the similarly named lines in Fig. 3, and represent the respective scales on the ruler. If T

be the point on the scale DG corresponding to the redemption price C, then from the similar triangles TDM and PNM

$$\mathrm{DM} : \mathrm{DT} = \mathrm{MN} : \mathrm{PN}.$$

If we suppose, as before, that $\mathrm{PN} = l/a_{\overline{n}|}^{(2)}$, and $\mathrm{DT} = m\mathrm{C}$,

then $$\mathrm{DM} = \frac{m}{l} \times \mathrm{C}(g-i)a_{\overline{n}|}^{(2)} = \frac{m}{l} \times k,$$

so that DM is proportional to the premium or discount.

It will be found convenient in practice to take the horizontal and vertical scales of the diagram in the ratio 10 : 1, in order to reduce it to manageable dimensions. In other words we take $l = 0{\cdot}1$. In this case the scale used on the ruler for discount or premium must also be ten times that used for the redemption price, but there is no need for them to bear any particular ratio to the scales used for the diagram.

The graphs are hardly distinguishable from straight lines for practical values of i, and after points have been plotted for multiples of 1 per-cent up to about 20 years, and perhaps multiples of $\frac{1}{2}$ per-cent thereafter, they may be joined with a ruler.

It is unnecessary to include the origin in the diagram, and the graphs may be drawn between two convenient limiting rates of interest. In these days gross rates of yield may be very high, while net rates of dividend, after deducting income tax, may be very low, so that a considerable range is required.

Graphs may be drawn for every half-year for the lower values of n, and at progressively increasing intervals thereafter, for the curves get closer together as n increases. If no graph has been drawn for the particular value of n required, a rough interpolation can be made by the eye. For small values of n the graphs are wide apart, and it may be found necessary to omit the lowest values, *e.g.*, $\frac{1}{2}$ and 1 year.

On a diagram drawn to the scales

$\frac{1}{8}$ inch $= {\cdot}0005$ horizontally

and $= {\cdot}005$ vertically,

which the writer has used constantly for many years, the yield can with care be read off with an error that seldom exceeds one penny per-cent. Used rapidly it can probably be trusted to the nearest multiple of 3*d*. per-cent, which is amply

sufficient for ordinary purposes. A smaller scale, in fact, might well be used.

A good scale for use with paper divided into tenths of an inch would be :—

for the diagram :— $\frac{1}{10}$ inch = ·0005 horizontally
and = 005 vertically.

for the ruler :—10 premium or discount, or 100 redemption price = 2 inches.

A diagram for rates of interest ranging from 2 to 10 per-cent, and for terms of $1\frac{1}{2}$ years and upwards would then measure 16 inches square, and the ruler could be made to read up to 40 premium or discount without being inconveniently long.

The advantages of the method, when great accuracy is not required, are the rapidity with which it can be used, the absence of tedious computations when a large number of results are required, and its equal applicability to gross and net rates, and to securities redeemable at par or otherwise.

The Valuation of Victory Bonds held to pay Estate Duty.

The Victory Loan is repayable at par, with interest at 2 per-cent half-yearly, by means of a half-yearly annuity of $2\frac{1}{4}$ per-cent on the original amount of the loan. The sum available each year for repayment of principal is to be applied in redemption of bonds by annual drawings at the end of the year.

If the drawings were half-yearly, the term of the loan would be almost exactly $55\frac{1}{2}$ years. Under the actual conditions the operation of the sinking fund will depend to a small extent upon how much interest is earned in the second half of the year on the balance of the first half-year's annuity after payment of interest. It is impossible therefore to state precisely the amounts that will be drawn for repayment each year. For practical purposes it may be sufficiently accurate to assume that the amounts outstanding at the ends of 1, 2, 3, &c., years, will be in the proportions $a_{\overline{55}|}$, $a_{\overline{54}|}$, $a_{\overline{53}|}$, &c., calculated at 4 per-cent.

On that assumption the present value—say, $a_{x\text{VB}}$—at rate i

of an annuity on the joint existence of a person aged x and a bond will be at date of issue

$$\{vp_x a^{4\%}_{\overline{55}|} + v^2{}_2p_x a^{4\%}_{\overline{54}|} + \ldots + v^{55}{}_{55}p_x a^{4\%}_{\overline{1}|}\}/a^{4\%}_{\overline{56}|}$$

or

$$\{a_{x\overline{55}|} - 1{\cdot}04^{-56}a'_{x\overline{55}|}\}/\{1 - 1{\cdot}04^{-56}\}*$$

and after t years

$$\{vp_x a^{4\%}_{\overline{55-t}|} + \ldots + v^{55-t}{}_{55-t}p_x a^{4\%}_{\overline{1}|}\}/a^{4\%}_{\overline{56-t}|}$$

or

$$\{a_{x\overline{55-t}|} - 1{\cdot}04^{-(56-t)}a'_{x\overline{55-t}|}\}/\{1 - 1{\cdot}04^{-(56-t)}\}$$

where a' is at the rate corresponding to $v' = 1{\cdot}04v$.

If $i = {\cdot}04$, a' becomes e. Also in many cases the temporary annuity becomes a whole-life annuity.

The following are a few examples of the value of $a_{x\text{VB}}$ at date of issue:

x	4%	5%	6%
50	11·694	10·666	...
55	10·440	9·607	...
60	9·071	8·433	...
65	7·639	7·170	...
70	6·207	5·882	5·589
75	4·847	4·634	4·441
80	3·621	3·492	3·371
85	2·574	2·501	2 431

On the assumption that the bondholder will be liable throughout for income-tax at 6*s*. in the £, the net half-yearly interest on the bonds is 1·4 per-cent, of which the yearly equivalent is 2·828 at 4 per-cent, 2·835 at 5 per-cent, or 2·842 at 6 per-cent.

Hence, if it be assumed that a bond will be applied to pay estate duty (if not previously redeemed) at the end of the year of death, its net 4 per-cent value at date of issue to a subscriber aged 50 is $2{\cdot}828a_{50\,.\,\text{VB}} + 102{\cdot}828\text{A}_{50\,.\,\text{VB}}$ at 4 per-cent,

* Mr. Lidstone, who independently investigated the problem on similar lines, points out that the result may be put into the following form, where j = the rate of interest on the bond and z the initial annual rate of the sinking fund:—

$$a - \frac{z}{j}(a' - a) \text{ or } a\left(1 + \frac{z}{j}\right) - \frac{z}{j}a'.$$

In the present case $j = {\cdot}04$; $z = {\cdot}005$, and the result at the outset is $1{\cdot}125a - {\cdot}125a'$.—Eds. *J.I.A.*

which $=85 \cdot 7$. Similarly the net 6 per-cent value at date of issue to a subscriber aged 75 is 83·8.

Five years after the date of issue it will be found that the net 4 per-cent value to a holder then aged 55 will be 87·3.

The following examples of approximate net yields at date of issue—on the basis of 6*s*. tax—may be of interest:

Age	Net Yield	Age	Net Yield
50	4·07	65	4·73
55	4·23	70	5·14
60	4·45	75	5·72

REVIEWS.

Pensions for Hospital Officers and Staffs: Report of a Sub-Committee of the Executive Committee of King Edward's Hospital Fund for London.

[C. & E. Layton and Geo. Barber. 1919. 7*s*. 6*d*.]

In April, 1914, the Executive Committee of King Edward's Hospital Fund for London appointed a Special Committee, consisting of Mr. W. J. H. Whittall, Sir William J. Collins, and Mr. Henry L. Hopkinson, to enquire into the question of pensions for Hospital Officers, and the result of their investigation has now been published in a Report which contains much that is of considerable interest from an actuarial point of view.

The first part of the Report deals with the existing provisions for pensions at London voluntary hospitals to which it is hardly necessary to refer. The second part gives a valuable account of the various ways in which pension funds can be arranged, with a number of examples showing how these plans have worked in practice. This part of the Report is largely historical, but we do not remember to have seen a clearer statement in general terms of the various schemes that have been adopted in one form or another; it affords a most helpful introduction to anyone who may wish to make a study of the subject.

With the parts of the Report mentioned above all the members of the Committee were in agreement; but when it came to the choice of a method suitable to the particular case, Mr. Whittall and Mr. Hopkinson recommended that pensions should be obtained through insurance companies, while Sir William Collins preferred a Pension Scheme arranged on a system by which a fund was to be set up for the officers and staffs concerned.

The problem which the members of the Committee had to face

was by no means an easy one, because hospital officers are employed by various authorities under different management, and it was advisable to make arrangements to facilitate transfer from one body to another without loss of pension rights and to make this arrangement so that while the pension is adequately guaranteed either by an insurance office or by some central fund, there was no interference with the employer's authority. But apart from this special difficulty, pension schemes of any kind are sufficiently difficult. If the pension be fixed it can be provided by regular contribution until the pension age is reached ; or it can be left for consideration when it actually has to be paid ; but even although it is comparatively easy to be convinced that the former is the better plan, we are always faced with the difficulty of providing for the pensions that have accrued in respect of past service, and even if we have surmounted this difficulty there is the doubt whether the assumed scale of salary will not be upset by a sudden alteration in salaries, such as has arisen recently owing to increased remuneration as the result of the greater cost of living consequent upon the war. There is also the difficulty in the particular case that the scales of salary by different employing bodies may differ considerably, and it needs but little imagination to see the many difficulties that would arise subsequently if a fund had been set up on such a basis. In the circumstances, therefore, probably every actuary would have agreed that one or other of the courses proposed by the members of the Committee should be adopted.

Perhaps in some respects the simplest plan is to arrange for the pensions through Insurance Offices ; but there are many objections, and although they are in no way shirked in the Majority Report, it is well that they should be explained. The first obvious difficulty is with regard to a choice of the kind of insurance that should be effected, and apparently the scheme which was particularly in the minds of Mr. Whittall and Mr. Hopkinson was that of the Federated Superannuation System for Universities. This system cannot be described as simple. It is arranged with a number of offices and by various kinds of policies ; it necessitates a number of increment policies from time to time as salaries are altered, and annuity rates are guaranteed in respect of endowment assurances when the policies mature ; so that the scheme, while pleasing in its variety, leaves the same feeling of bewilderment as results from attempting to decide upon a suitable form of insurance from an overcrowded prospectus of a Life Insurance Company. We must confess that we have very little sympathy with such complications and there is no need to link them with the purchase of pensions from insurance companies. As the object of a pension scheme is to provide pensions, and as a member of a staff can provide his own insurance through the usual channel the simplest arrangement is to provide pensions by means of deferred annuities, either with or without return of contributions in the event of death. Even such an arrangement, however, leaves out of account the difficulty of making proper provision for pensions in the event of early disablement. If, for instance, an

officer breaks down at age 40, some form of pension has to be provided, and no insurance scheme that has yet been devised will help. Mr. F. L. Collins, who gave some interesting information to the Committee, showed how by "nursing" insurance policies it would be possible for a central fund to make provision in this respect. But if a central fund has to be set up for such a purpose, one is naturally turned in the direction of Sir William Collins' dissentient opinion, that it would be best to use a central fund only and not go to the insurance companies at all.

We can now turn to what is, from the actuarial point of view, perhaps the most interesting part of the work. This is Mr. T. Tinner's suggestion of a "money purchase scheme", which is to provide pensions not only at the retiring age, but also on previous invalidity. The underlying theory is that a sum of £1 paid at any age will provide a pension of £p a year at the pension age or earlier retirement, and it shows how the pensions compare with those that are reached by the ordinary salary scale method. Broadly speaking, he found that the retiring allowance on early invalidity was somewhat greater than would be given under the ordinary salary scale arrangement of the old civil service scheme (a sixtieth of final salary for each year of service, with a maximum of 40-sixtieths). He also found that those who came in at a very early age and continued to the latest age at which they can be pensioned ought to receive a larger pension than that paid under the salary scale system. The advantages of such schemes are that they avoid all the difficulties connected with salary scales, and are therefore far less likely to lead into insolvency; they get round all the difficulties that result from a number of employing bodies, and although it is not easy to set out the particulars in the form which conveys quickly the pension to which a person would be entitled, it would be easy to convince any contributor that each payment purchases a reasonable amount of pension. The objections are that people are not accustomed to "money-purchase systems", and prefer pensions to be fixed in amount or fixed as a proportion of salary. Probably similar systems have been suggested by other actuaries, and in the simple form without the invalidity benefit it has of course been adopted in many cases; but we feel that considerable credit is due to Mr. Tinner for the way he has set out the scheme, and it is to be hoped that at some future date he may give a somewhat fuller discussion of his data and methods.

Although there is a Main Report and a Dissentient Memorandum, we are inclined to think that there is not a grave difference of opinion between the various members of the Committee. If we read the Report correctly, Mr. Whittall and Mr. Hopkinson reject the mutual principle largely because they feel that there is an absence of financial guarantee which the hospitals themselves cannot give and which King Edward's Hospital Fund may be either unwilling or unable to provide. The reader will find both sets of views so well expressed that his sympathies will in turn be with everyone concerned.

The whole Report is well worth reading. It has the merit of

being well printed and well set out, and the difference of opinion between the members of the Committee, although it was no doubt not what anyone of them would have wished, adds—it must be confessed—a little zest to the reading.

W. P. E.

Actuarial Studies: No. 1, *Sources and Characteristics of the Principal Mortality Tables. Pp.* 79. *No.* 4, *Graduation of Mortality and other Tables. Pp.* 82.

[Actuarial Society of America: 346, Broadway, New York. 1919.]

THE publication of this series of essays is an event of considerable importance in the history of actuarial education, and its progress will be followed with much interest by all English-reading actuaries. Being written by American actuaries primarily for American students, the essays will no doubt present certain differences, in point of view and treatment, from any similar series which might be compiled in the United Kingdom—differences which will probably be more marked in the essays dealing with the practical application of actuarial principles to life assurance business (with valuations and the distribution of surplus, for example) and to social insurance than in those of a more theoretical character—but they must necessarily cover a good part of the ground that would be covered by a Text-Book, Part III.

In No. 1 of the series Mr. Henry Moir and his associated contributors have written as interesting an essay as could reasonably be expected on a rather dull subject. The short accounts of Dr. Sprague's Select Tables (in which four original features are specially commended, namely, the use of a common radix, the junction with the $H^{M(5)}$ Table, the introduction of osculatory interpolation, and the development of a scheme of notation) and of the British Offices' Tables are good and useful. But the sections to which students in this country will turn with most interest are those dealing with the various American Tables. It is rather remarkable, in view of all the work that American actuaries have done on mortality statistics, that there should have been no U.S. population tables worth mentioning until the 1910 Table, and no select tables until the M.A. In connection with the latter table the statement that being based on policies "it should not be applied to "the solution of financial problems since mortality rates are higher "when based upon amounts insured" seems to us obscure; there are many financial problems to which a table based on policies is at least as applicable as one based on amounts. Another incidental comment that appears open to question is the criticism that in the British Offices Experience the inclusion of the "Old Assurances" had the effect of introducing an increased number of non-select and aged lives, and consequently of making the aggregate mortality curve "unduly steep" after middle life; the true ground of objection (if any) to the inclusion of the "Old Assurances" would seem to be

rather that they belonged to an earlier generation, and that the Experience as a whole would have been more homogeneous if it had been restricted to the "New Assurances."

No. 4, to which Mr. Robert Henderson is the principal contributor, deals with a subject of more academic interest and admitting of more originality of treatment. It is perhaps somewhat too condensed for a text-book, but it will undoubtedly be very helpful to the best students—whether before or after their examinations—and to teachers. The general method of treatment is as follows: After a short introduction on the reasons for graduation, &c., the four classes of methods—graphic, interpolation, summation, and mathematical—are successively discussed, each being applied incidentally for purposes of illustration to the graduation of a limited and very irregular experience, namely, the $O^{F(50)}$, and in conclusion the four graduations are compared in respect of smoothness and agreement with the data. The graduations and comparison should of course be understood by the student to be entirely of the nature of an example. Clearly no method of graduation can discover the true law of such scanty data, and in the absence of *à priori* evidence in favour of some specified law such as Makeham's it would probably be best in practice to graduate with reference to some similar data of greater extent.

In the application of the interpolation method to the illustrative experience (p. 22), a 3rd-difference formula including five values of w (*i.e.*, 25-terms of the original series)* is used for determining the pivots, instead of a strict interpolation formula. The precise nature of the course adopted might have been explained with advantage, since it involves a material departure from Mr. King's method as defined on p. 7, and as described on pp. 18–21, and also because it introduces the important principle of the reduction of mean square error. American students will probably be familiar with Mr. Henderson's "Mortality Laws and Statistics" (to which, however, no references are given), but by other readers the point of

* Let $u_0 = pw_0 + qw_{\pm 5} + rw_{\pm 10}$, where $w_x = u_{x-2} + \ldots + u_{x+2}$.

(1) Assuming 5th differences of u to be constant we obtain Mr. King's interpolation formula iii*a* (*J.I.A.*, vol. xliii, p. 114).

(2) Assuming 3rd differences to be constant we can determine p, q, r, so as to satisfy one additional condition.

(*a*) Making the sum of the squares of the coefficients a minimum, we obtain the "best" value of u, namely,

$$(696w_0 + 488w_{\pm 5} - 136w_{\pm 10})/7000.$$

R^2 (the reduction of mean square error) $= \cdot 1018$.

The minimum value of R^2 for a range of 25 terms is $\cdot 090$ (*J.I.A.*, vol. xlviii, p. 407).

(*b*) Putting $p = q$, we obtain the formula used by Mr. Henderson $(51w_0 + 51w_{\pm 5} - 14w_{\pm 10})/625$

$R^2 = \cdot 1050$.

(*c*) Putting $r = 0$, we obtain Mr. King's interpolation formula $\ddot{\text{V}}a$

$R^2 = \cdot 2340$.

the formula may well be missed. The formula is unquestionably a good formula of its kind, but if a combination of the adjusted-average and interpolation methods is to be employed there would seem to be no reason why the best adjusted-average formula should not be used, nor why the restriction of an initial summation in fives should not be removed when the figures for individual ages are available. Possibly this may be contemplated in the brief further reference to the subject on page 76.

Summation-formulas are dealt with effectively by means of a powerful scheme of symbolical operators (in a future edition the use of E to denote both an error and an operator might be avoided, especially as it occurs in both senses in the same paragraph). The general term is given for the 3rd difference formula of specified range and maximum smoothing power, and the reader is referred to Mr. Henderson's T.A.S.A. paper on Graduation by Adjusted Average. The general proposition is not, however, easy—nor of any importance to the great majority of students—and it would perhaps have been more instructive to give a simple example of finding the formula with maximum smoothing-power for a given combination of operators, *e.g.*, three 5's, and a 5-term operand. We should like to have seen Dr. Sheppard's tables of maximum reductions of error and minimum smoothing-coefficients included. These tables form convenient standards by which summation formulas may be tested.

In the sample Makeham graduation with an arbitrary value of c, a trial-and-error method is employed for determining the other constants so as to reproduce exactly the total number of deaths and the first moment. The method is simple in application, but the theoretical explanation is rendered somewhat obscure by the introduction (apparently for neatness and convenience) of an intermediary constant which is not absolutely necessary and by a misprint of a for m in the first line of p. 62.

We have selected for comment the few points that suggest criticism rather than the many that might justly claim commendation, but we may close with a word of appreciation of the comparison of the four sample graduations. It is a comparison of graduations rather than methods since all four graduations (even those that are sometimes called "mechanical") are markedly individual—they form indeed an object-lesson showing that any method of graduation leaves something to the personal judgment and skill of the graduator—but as an example of the way in which the tests of smoothness and agreement are applied it is most instructive. One other feature which must be specially commended is the inclusion of a useful (although not complete) list of authorities and papers on the subject of graduation.

THE INSTITUTE OF ACTUARIES.

EXAMINATIONS, JUNE 1919.

BOARD OF EXAMINERS.

Examination for Admission to the Class of Associate (Part I.—Section A).

First Paper.

1. A sets out to walk from London to Brighton at the same moment that B sets out to walk from Brighton to London. Each walks at a uniform pace. The time that A takes to reach Brighton from the point where they meet is 21 per-cent longer than the time that B takes to reach London from the same point. How much per-cent longer is A over the whole journey than B?

2. Find the condition that the roots of a quadratic equation may be real.

Show, without the use of the differential calculus, that the value of the expression $\dfrac{x^2-14x+5}{(x+3)^2}$ can never be less than $-\dfrac{11}{14}$, when x is real.

3. Solve the equations

(1) $x^4-5x^3+8x^2-5x+1=0$

(2) $x^2+3x+2\sqrt{3x^2+9x+7}=0$

(3) $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{xy}+\dfrac{1}{yz}+\dfrac{1}{xz}=x+y+z=7$

4. Find the values of A which will make the expression

$$8x^2-3y^2-15z^2+10xy+\text{A}yz-14xz$$

represent the product of two rational factors.

5. If you were required to find the sum of the following series, how would you determine by inspection in each case whether to attempt a solution in the first instance by

(1) regarding the series as derived from the binomial series
or (2) ,, ,, ,, exponential ,,
or (3) ,, ,, ,, logarithmic ,,
or (4) employing the method of differences?

(*a*) $\frac{1}{3.4.5}+\frac{1}{5.6.7}+\frac{1}{7.8.9}+\ldots$ to ∞

(*b*) $\frac{1}{4.6}+\frac{1.3}{4.6.8}+\frac{1.3.5}{4.6.8.10}+\ldots$ to ∞

(*c*) $\frac{3}{7}+\frac{3.5}{7.9}+\frac{3.5.7}{7.9.11}+\ldots$ to ∞

(*d*) $\frac{1^2}{3.4}+\frac{1^2+2^2}{3.4.5}+\frac{1^2+2^2+3^2}{3.4.5.6}+\ldots$ to ∞

The reasons for your decision should be given, but no calculations are required.

6. Show that for certain values of x

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

and state, without proof, for what values of x this identity holds.

Given $\log_{10}e=\cdot 43429$ calculate $\log_{10}85$ correct to four places.

7. If a number of five figures containing any five of the ten digits once only is written down at random, what is the probability that it is divisible by 9?

8. The 26 letters of the alphabet are placed in a bag. A and B alternately draw a letter from the bag, the letters drawn not being replaced. The winner is the one who draws most vowels. A starts and draws a vowel with his first draw. What is his chance of winning?

9. A and B play a set of games, to be won by the player who first wins four games, with the condition that if they each win three they are to play the best of three to decide the set. A's chance of winning a single game is to B's as 2 to 1. Find their respective chances of winning the set.

Second Paper.

1. $u_1=1$; $u_2+u_3=5\cdot 41$;
$u_4+u_5+u_6=18\cdot 47$;
$u_7+u_8+u_9+u_{10}+u_{11}+u_{12}=90\cdot 36$.

Find the value of u_x for all values of x from 1 to 12 inclusive

2. The following values of a certain function u are supplied to you. Which, if any, of these values would you consider affected by a misprint, and how would you, with the material in the question, supply a more correct value?

t	u_t
1	21·4
2	14·9
3	11·4
4	7·8
5	5 8

3. Given
$$u_0 = 58{\cdot}842$$
$$u_2 = 55{\cdot}257$$
$$u_4 = 51{\cdot}368$$
$$u_{10} = 37{\cdot}977$$
complete the series $u_0, u_1, u_2 \ldots u_{10}$.

4. Prove that $\dfrac{du_x}{dx} = \dfrac{u_{x+m} - u_{x-m}}{2m}$ approximately.

Give a geometrical interpretation of this approximation.

5. Find from first principles the differential coefficient of x^n.

Find the differential coefficients with regard to x of

$$\log x^{\frac{1}{x}}$$

$$\frac{\sqrt{x^2 - 3}}{(x-1)\sqrt{x^2 - 7}}$$

and with regard to x^2 of

$$\frac{\sqrt{x^2 - 2}}{x}$$

6. State and prove Leibnitz' theorem as to the nth differential coefficient of the product of two functions.

Prove that

$$\frac{\log (x + \sqrt{1+x^2})}{\sqrt{1+x^2}} = x - \frac{2}{3}x^3 + \frac{2 \,.\, 4}{3 \,.\, 5}x^5 - \ldots$$

7. A man in a boat at sea, 5 miles distant from the nearest point of a straight shore, wishes to reach a place 12 miles distant along the shore, measuring from this nearest point.

At what point should he land to reach this place in the minimum time, if he can row at 3 miles an hour and walk at 4 miles an hour?

8. Evaluate (i) $\int e^x x^3 . dx$

(ii) $\int_1^2 \frac{dx}{x^2\sqrt{2+x^2}}$

(iii) $\int \frac{dx}{x^2+3x-4}$

9. Prove that $\int_3^5 f(x)dx = \frac{1}{12}\{f(2)+22f(4)+f(6)\}$ approximately, and hence find an approximate value for $\int_{1\cdot 5}^{2\cdot 5} 2^{-x^2}dx$.

Examination for Admission to the Class of Associate (Part I.—Section B).

1. Explain what is meant by "nominal" and "effective" rates of interest.

If £100 amounts to £112·73 in 3 years at a nominal rate of interest of 4 per-cent per annum, how often is interest convertible?

Given $\log_{10} e = \cdot 434294$

$\log 1\cdot 1273 = \cdot 052029.$

*2. Given $3a_{\overline{n}|} = 2a_{\overline{2n}|} = 45$, find n and i.

*3. A loan of £10,000 due 30 June 1919 is to be repaid by an annuity in $10\frac{1}{2}$ years, a half payment being made on 31 December 1919 and a full payment at the end of each of the succeeding ten years. Find the amount of the annuity at 4 per-cent per annum and draw up a schedule showing the division of each payment into principal and interest.

*4. On 1 January 1919 an investor gave £1,200 for an annuity of £100 payable annually from 1 July 1919 to 1 July 1936 inclusive. If he is to earn $4\frac{1}{2}$ per-cent per annum on his whole capital throughout the period of the transaction, at what rate of interest must he accumulate the sinking fund to replace his capital at the end of the period?

*5. The following 4 per-cent loans have been borrowed:

(1) £10,000 on 1 July 1909, repayable in 20 years by means of a level yearly annuity.

(2) £5,000 on 1 January 1913, repayable in 25 years by means of equal half-yearly instalments of principal, with interest payable half-yearly on the balances outstanding.

It is proposed to consolidate these loans into one, as from 1 July 1919, repayable in 14 years at 4 per-cent per annum by means of a level half-yearly annuity.

What will be the future half-yearly payments?

*6. A government purchases a railway under the terms of an agreement whereby it must pay £137 for every £100 of ordinary stock. As an alternative to payment in cash it offers to each holder of £100 stock either:

(*a*) A terminable annuity of £7. 13*s*. 8*d*. per annum payable half-yearly for 40 years, or

(*b*) £120 irredeemable stock bearing interest at 4 per-cent per annum payable half-yearly, together with a triennial dividend based on the net profits of the railway.

What rate of dividend must be paid under (*b*) in order that the yields under (*a*) and (*b*) may be equal?

*7. A £20 share is bought for £50; the dividends are payable half-yearly and for the first year are declared at the rate of 10 per-cent per annum, for the second year at $9\frac{3}{4}$, the third year at $9\frac{1}{2}$, decreasing by equal decrements annually to $7\frac{1}{2}$ per-cent per annum for the eleventh year.

At the end of the eleventh year the share is sold for £45. What yield has the purchaser obtained on his investment?

* "A Short Collection of Actuarial Tables" will be supplied for use in answering these questions.

Examination for Admission to the Class of Associate

(Part II).

First Paper.

1. Given that out of 1,000 children at age 0, the deaths under three months old are 40, between three and six months 28, and between six and twelve months 36, find L_0, m_0 and $\mu_{\frac{1}{2}}$ as accurately as possible. Explain why $\mu_{\frac{1}{2}}$ is less than m_0.

2. If the probability that exactly three lives out of six all aged x survive n years is ·08192, find the probability that at least three survive n years.

3. In the case of three lives all aged x find

(1) The probability that the first death will occur in the nth year;

(2) The probability that the second death will occur in the nth year;

(3) The probability that the third death will occur in the nth year.

4. A staff, including both those in active service and those who have been pensioned, is represented, just after a year's entrants have joined the staff and a year's retirements have taken place, by the l_x column from age 20 onwards, and has reached a stationary condition.

Retirement on pension takes place at exact ages 61 to 65, and there are no withdrawals except by death. The proportion of those in active service at each age who retire at that age and the corresponding pensions are as follows:

Age	Proportion of those in active service at each age who then retire	Amount of Pension (assume payable momently)
61	One-tenth	£140
62	One-sixth	£160
63	One-third	£180
64	One-half	£200
65	All	£220

Assuming that the mortality among pensioners is the same as among those in active service at corresponding ages, find expressions for the following, just after a year's retirements have taken place:

(1) The total number of pensioners.

(2) The total pensions payable in a year.

(3) The sum (ignoring interest) of all future payments in respect of existing pensions.

*5. A widows' annuity-society pays an annuity of £10 per annum, the first payment being due at the end of the year in which the husband dies. The society is recruited at the beginning of each year by the entrance of 100 married couples, the husband being aged 30 and the wife 25. After five years from the formation of the society, how many widows will there be on the fund, and what will be the value of the pensions then payable?

Use H^M functions and value at 3 per-cent interest.

6. If μ_x is in the form of $A + Bc^x\left(\log_e x + \frac{1}{\log_e c}\cdot\frac{1}{x}\right)$, find an expression for l_x.

7. Explain how you would calculate the annual premium for an assurance payable at the death of x if he dies after the survivor of y and z but before w.

* "A Short Collection of Actuarial Tables" will be supplied for use in answering this question.

Second Paper.

1. Explain the advantages of commutation columns.

Express in commutation symbols the annual premium for a decreasing term assurance for m years, the sum assured in the event of death in the nth year (n not greater than m) being of the form

$$B(m-n)+C$$

and the premium for any year being proportionate to the sum assured for that year.

2. You are asked by a client to quote the present value of a sum of £10,000 receivable immediately upon the death of a man aged 60 on the basis of H^M Mortality and 5 per-cent interest. Your client has referred to a table of complete expectations of life and has calculated the value himself by assuming 5 per-cent compound interest for the expectation as a term certain.

Set forth in language as little technical as possible the reasons you would give him to account for the difference between your value and his. Show clearly the systematic nature of the difference, and state in which direction the error will prevail.

3. Explain what you understand by the Law of Uniform Seniority.

Does this law apply to a table for which $\mu_x = A + Bx$?

4. If $a_{x:\overline{n}|} = \alpha - \beta c^x$, expand $A_{x:\overline{n}|}$ in ascending powers of x.

If you were given a complete table of $a_{x:n}$ indicate how you would deduce the constants to be employed in the calculation of $A_{50:\overline{15}|}$.

5. Find, in terms as far as possible of contingent survivorship assurances payable on y dying first, the single premium for 1 payable

(*a*) On the death of y within 10 years, x surviving him and both u and v having died previously;

(*b*) On the death of y after 10 years, x surviving him, one at least of u and v having died previously, and all having survived 10 years;

(*c*) On the death of y after 10 years, x surviving him and one only of u and v having died previously.

6. Find the value at 3 per-cent interest of a whole life policy for £1,250 effected at age 30 which has been 20 years in force and to which a reversionary bonus of £500 is attached, having given that $P_{30} = \cdot 0179$ and $P_{50} = \cdot 0373$.

7. How would you verify a table of endowment assurance policy-values by addition?

Third Paper.

*1. If the net single premium for a whole life policy for £100 at a given age is £32. 8*s*., the corresponding net level yearly premium £1. 12*s*. 5*d*., and the corresponding net yearly premium reducible by 50 per-cent after 10 years £2. 5*s*. 8*d*., find the net single premium on the same basis for a 10-year endowment assurance for £100.

*2. A whole life assurance for £500, subject to a yearly premium, was effected on a life aged 20, 10 years ago. Five years ago, just before payment of the sixth premium, it was altered to an endowment assurance at age 60, an increased annual premium being paid in consequence of the alteration. Find the value of the policy at the present time. Use the $O^{[NM]}$ 3 per-cent Table throughout and ignore the question of loading.

*3. An office has on its books at the beginning of a year 100 annuities of £100 on lives aged 70, each payable by quarterly instalments on 31 March, &c., with proportion to date of death. It makes reserves on the Carlisle 3 per-cent basis and earns 4 per-cent on its funds. If during the year there are four deaths amongst these annuitants find what profit or loss the office will make in the group for the year. Assume the deaths are evenly spread over the year and that there are no expenses.

*4. Find by the H^M 3 per-cent Table the net single premium for an assurance of 1 payable on the attainment of age 21 by a child now aged 7, or, if this child dies previously, on the attainment of age 21 by a child now aged 5, or, if neither attain 21, then on the death of the survivor of them. Given that ${}_{\overline{14|}}A_{5:7} = \cdot 101$.

*5. A policy for £1,000, under which the sum assured is payable at death after age 21, the premiums paid being returnable with 3 per-cent compound interest at death before age 21, is granted at an annual premium on the life of a child aged 1. It is desired that the option shall be granted at age 21 without alteration in the annual premium to convert the policy either (1) to an endowment assurance for a reduced amount payable at age 55 or previous death, or (2) to an assurance for a reduced amount payable at death with the number of future premiums (including the one at age 21) limited to a maximum of 30. Assuming H^M Mortality with 3 per-cent interest find the reduced sum assured in each case.

*6. An under-average life aged 50 takes out an endowment assurance with profits, for a term of 15 years. He is offered two alternatives,

(1) to pay an extra annual premium of 10*s.* per-cent; or

(2) to pay the normal premium, but the bonus to be payable to him only if he should survive the term.

What rate of annual simple reversionary bonus will make these options equal in value to one another?

Assume that the life can be treated throughout with an addition of five years to the age, and work with the $O^{[NM]}$ 3 per-cent Table.

7. On the death of the survivor of x and y a sum of 1 is to be divided equally among such of three persons aged a, b and c respectively, as may then be living, with a further condition that if a predecease the survivor of x and y his share is to be divided equally between such of two persons aged u and v as are living at the date of distribution.

Express in the form of an integral the value of u's share.

* "A Short Collection of Actuarial Tables" will be supplied for use in answering these questions.

Examination for Admission to the Class of Fellow (Part III.—Section A).

First Paper.

1. Given the following particulars, how would you deduce the "exposed to risk" for the purpose of (a) an aggregate and (b) a select mortality experience of a life assurance society?

Calendar year of entry. Age next birthday at entry.

Calendar year of exit. Mode of exit and due date of last premium paid.

Assume that the investigation is to cover a period of 15 years and that all premiums are annual.

2. Work out the rates of mortality that can be obtained conveniently from the following information and state what tables in this form would be required to obtain complete select tables for durations 0 to 10 and ultimate tables thereafter:

Number of policies on the books which, at the date mentioned, were more than 8 *and less than* 9 *years in force.*

Date	Age last Birthday	
	60	61
1 Jan. 1918	3,000	2,850
1 Jan. 1919	3,400	3,300

Number of deaths during 1918, *under policies which at date of death were more than* 8 *and less than* 9 *years in force.*

Age last Birthday at Death	Number of Deaths
60	100
61	110

3. State shortly the various steps taken in forming the English Life Table, No. 7.

4. The following table has been prepared from the Census Returns of 1901 and 1911 by applying the rates of mortality shown by the English Life Table No. 7 to the population groups at the earlier census. Analyze and explain the divergencies in these figures :

Ages at 1911	Males		Females	
	Number (in thousands) expected to have survived from 1901	Number (in thousands) enumerated	Number (in thousands) expected to have survived from 1901	Number (in thousands) enumerated
15–19	1,698	1,655	1,706	1,682
20–24	1,620	1,503	1,624	1,673
25–29	1,543	1,456	1,582	1,623
30–34	1,400	1,376	1,579	1,501
35–39	1,246	1,261	1,417	1,352
40–44	1,067	1,075	1,190	1,158
45–49	930	926	1,020	999
50–54	779	768	853	834
55–59	625	608	699	670
60–64	483	477	560	543
65–69	337	366	410	441
70–74	234	237	305	317

5. Describe shortly the methods employed in the graduation of the Whole Life Without Profit Mortality Table (British Offices Life Tables 1893).

At what age at entry is the select table started and what is the general effect of the graduation on the mortality of the first five years of assurance?

6. What are the tests of a good graduation?

Second Paper.

1. The following facts are available:

(*a*) Number of persons at each age last birthday on 1 January of each year.

(*b*) Number of persons at each age last birthday marrying, dying or withdrawing during each calendar year.

(*c*) Number of entrants each year with dates of birth.

It is suggested that rates of death, marriage and withdrawal should be obtained by dividing the number of deaths, marriages and withdrawals for each age by the sum of the numbers at each 1 January, decreased by half the deaths and increased by half the entrants (using age last birthday at entrance). Explain whether this is satisfactory, and if unsatisfactory suggest improvements.

2. What are the special objects sought in constructing a mortality table:

(*a*) for general use in connection with life assurance work?

(*b*) for the purpose of comparative statistics?

3. Explain clearly the method you would adopt in investigating the sickness experience of a large friendly society, and deducing rates of sickness under headings "1st six months," "2nd six months," "after first year."

What inaccuracy is involved in the method you use, apart from that arising from the way in which you arrive at the age?

4. The average deviation, irrespective of sign, is sometimes given as approximately $\cdot 8\sqrt{npq}$. How is this function used in actuarial work, what are the assumptions underlying its derivation, and how far are these justified in the case of mortality statistics?

*5. Discuss the graduation shown in the following table. Set out the result graphically for ages 20–50, and show on your diagram any improvement you consider necessary.

Age x	Exposed to risk from $x-2$ to $x+2$	Force of Mortality at x		Expectation of Life	
		Ungraduated	Graduated	Ungraduated	Graduated
10	429	·00233	·00350	47·84	47·60
15	669	·00299	·00355	43·40	43·32
20	990	·00404	·00456	39·13	39·16
25	4,400	·00609	·00585	34·96	35·12
30	14,000	·00715	·00757	30·94	31·22
35	30,000	·01090	·00986	27·25	27·47
40	40,000	·01440	·01225	23·87	23·92
45	60,000	·01795	·01713	20·65	20·58
50	70,000	·02327	·02295	17·60	17·46
55	70,000	·03067	·03100	14·76	14·60
60	70,000	·04125	·04245	12·14	12·00
65	60,000	·05624	·05875	9·76	9·69
70	40,000	·08000	·08150	7·66	7·70
75	20,000	·11636	·11500	5·91	6·00
80	10,000	·16760	·16272	4·53	4·59
85	2,000	·23615	·23078	3·48	3·46
90	200	·32500	·32669	2·69	2·58

You may assume that the deaths are given sufficiently accurately by the product of the figures in the columns headed "exposed to risk" and "force of mortality."

* Sheets of cross-ruled paper will be supplied for use in answering this question.

Examination for Admission to the Class of Fellow (Part III.—Section B).

First Paper.

1. Explain briefly the theoretical assumptions underlying Lidstone's Z method of valuing endowment assurances.

How would you apply the method to the case of an office having a large number of policies maturing on the actual birthday, with premiums payable for $n+1$ years, where n is the difference between the age next birthday at entry and the maturity age?

2. State concisely the different methods of valuing whole life policies by limited payments. Which method do you prefer, and why?

3. The question as to the advisability of making a change in the valuation basis is under consideration by your company.

To what extent and under what conditions would you be justified in making use of the tables that have been prepared, based on the "Model Office", showing the relative reserves according to different valuation tables and at different rates of interest? Mention some of the precautions and limitations that must be borne in mind.

Consider as examples:

(1) The change from O^M 3 per-cent to O^M and $O^{M(5)}$ 3 per-cent.

(2) The change from O^M $2\frac{1}{2}$ per-cent to O^M 3 per-cent.

4. What method would you adopt in valuing the liabilities of an old established employers' liability insurance company?

5. What considerations would guide you in determining the maximum sum assured to be retained by a life office at its own risk on any one life?

6. Contrast the relative position of with profit policyholders in two mutual offices A and B. The last valuation of each office took place as at 31 December 1914, and since that date both offices are known to have suffered a depreciation equal to 10 per-cent of the invested funds as at 31 December 1914. This depreciation will fall to be dealt with at the next valuation at the end of 1919.

	Office A	Office B
Valuation Basis	O^M $2\frac{1}{2}$ %	O^M 3 %
Percentage of Business, with Profits...	50 %	95 %
Percentage of Gross Premiums, reserved for expenses	15·	21·5
Assurances in force	£4,500,000	£5,000,000
Bonus last declared	30*s.* % per ann. compound	30*s.* % per ann. simple
Surplus carried forward	£50,000	Nil.
Funds 31 December 1914	£2,000,000	£2,000,000
" " 1917	£1,950,000	£2,400,000
Average annual expense ratio, 1915–1917	$12\frac{1}{2}$ %	14 %
Average annual rate of interest (1915–1917), after deducting tax	£3. 17*s.* 0*d.*	£4. 0*s.* 0*d.*
Percentage of actual to expected claims	90	70

Second Paper.

*1. Table I gives a summary of the particulars for valuation of a closed endowment assurance fund as at 31 December 1913.

Table II gives particulars of policies cancelled since that date.

Table III gives certain items of the consolidated revenue account for the five years ending 31 December 1918.

Table IV gives certain values of Z_M.

The fund is managed at an expense ratio of $12\frac{1}{2}$ per-cent of the premium income. At the 1913 valuation—on the O^M 3 per-cent basis—a simple reversionary bonus of 2 per-cent per ann. for the quinquennium was declared. £260 undivided surplus was carried forward and it was decided to pay an interim bonus of 30*s.* per-cent for each complete year's premium paid on policies becoming claims by death. The average net annual rate of interest earned 1914 to 1918 was £3. 10*s.* per-cent, and at 31 December 1918 it was found that the invested funds had depreciated in value 8 per-cent.

Complete the consolidated revenue account.

Summarize the existing business for valuation as at 31 December 1918.

Deduce the valuation ages. Make the valuation, using the tables of v^n and $a_{\overline{n}|}$, in the absence of tables of $A_{x.\overline{n}|}$ and $a_{x.\overline{n}|}$, at rates of interest respectively $\frac{1}{2}$ per-cent lower and 1 per-cent higher than the rate which you would adopt were O^M Tables available.

Draw up a profit and loss account and show what bonus you would recommend.

* "A Short Collection of Actuarial Tables" will be supplied for use in answering this question.

TABLE I.

Year of Maturity	Sum Assured	Existing Bonus	Office Annual Premium	Balance of Year's Premium outstanding at 31 December	Net Annual Premium	$Z_M \times$ Sum Assured
	£	£	£	£ s. d.	£	
1923	8,500	805	381	18 0 0	255	9,224
1928	7,000	480	292	...	211	6,635
1933	4,750	278	208	13 15 0	140	6,079

TABLE II.

Policy	A	B	C
Date of Entry ...	1 March 1898	1 Sept. 1908	1 Nov. 1903
„ Exit ...	1 February 1916	1 August 1916	1 August 1918
„ Birth ...	1 January 1873	1 Dec. 1875	1 Feb. 1873
„ Maturity	1 March 1923	1 Sept. 1928	1 Nov. 1933
Cause of Exit ...	Death	Death	Surrendered for £240
Sum Assured ...	£500	£1,000	£750
Bonus to 1913 ...	£150	£50	£75
Office Annual Premium ...	£21. 5*s*. 6*d*.	£52. 10*s*.	£27. 10*s*.
How payable and when due ...	Half-yearly on 1 Mar. & 1 Sept.	Yearly on 1 September	Half-yearly on 1 Nov. and 1 May
Net Annual Premium OM 3 per-cent ...	15·4	41·2	19·8

TABLE III.

Fund as at 1 January 1914	£9,600
Interest less Tax	1,900
Premium Income	4,172

TABLE IV.

M	Z_M	M	Z_M	M	Z_M
50	·681	54	·926	58	1·260
51	·735	55	1·000	59	1·360
52	·794	56	1·080	60	1·469
53	·857	57	1·166	61	1·587

2. A life insurance company charges premiums which, allowing for expenses, are exactly sufficient to provide for an uniform compound reversionary bonus. Simple reversionary bonuses are, however, declared up to the full surplus by a strong net premium valuation. What will be the effect on subsequent bonuses of a continuation of this practice, and how may the effect be accentuated or masked by new business?

3. An office has issued a very large number of short term endowment assurances, and wishes to make a separate valuation of them, and to keep a separate fund and accounts in respect of them, the fund having to bear the actual expenses incurred in connection with the business. Many of the premiums are payable by half-yearly or quarterly instalments, the unpaid balance of the year's premium, current at the time of death, being deducted from the policy moneys.

Explain in detail the method of valuation you would adopt.

4. It has been decided to amalgamate two mutual life offices by merging their business and assets, instead of by closing the fund of one office. Explain fully the principles which should be adopted in carrying out the fusion, with special reference to the method of distributing surplus:

(i) Up to the date of the amalgamation, and

(ii) Thereafter.

State your views as to the advantages or disadvantages of such an amalgamation as compared with a transfer involving the closing of the fund of the office transferred.

Examination for Admission to the Class of Fellow (Part IV.—Section A).

First Paper.

1. What are the regulations laid down in the Assurance Companies Act, 1909, governing the transfer of the life assurance business of one company to another?

2. What classes of societies may be registered as friendly societies under the Friendly Societies Act, 1896?

What limits exist as to the amounts of assurance or annuity which may be granted by a registered friendly society?

3. Define "industrial insurance company" and "collecting society." What restrictions exist in the case of contracts made by these bodies in respect of (*a*) Insurance on infants; (*b*) Forfeiture of policies?

4. Give a short account of the provisions of the Assurance Companies Act, 1909, with regard to making deposits for different classes of business. How do these provisions differ from those under the 1870 Act?

5. In what circumstances may a life assurance company be wound up by the Court?

State the principles laid down by the Assurance Companies Act, 1909, for valuing the assurance and annuity contracts of a life office in the event of winding up.

6. An office has on its books three classes of short term endowment assurances, under which the sum assured payable at death or maturity is payable respectively,

(*a*) In cash ;

(*b*) By the transfer of War Loan, 1929–1947 ;

(*c*) By the transfer of War Bonds, redeemable 1 October 1927.

What bases would you recommend for the surrender values to be allowed under (*a*), (*b*) and (*c*)?

7. What are the methods usually adopted by offices in calculating surrender values of:

(1) Sinking fund or leasehold redemption policies ;

(2) Simple endowments, with return of premiums at death before maturity ;

(3) Children's educational annuities, with return of premiums or part premiums at death of child within the annuity period?

and state why they sometimes differ as between the three classes of policy.

8. How would you calculate the surrender-values of participating policies converted from whole-life to endowment assurances?

State your views as to the desirability of endorsing on ordinary whole-life and endowment assurances tables of guaranteed surrender-values.

Second Paper.

*1. A, aged 40, and B, aged 44, are each entitled to a moiety of the undermentioned fund, contingently on their surviving their mother, aged 71. In the event of one only surviving, such survivor takes the whole fund, but if both predecease their mother, the estate passes elsewhere.

£10,000 5 per-cent War Loan, 1929–1947 ;

£5,000 $3\frac{1}{2}$ per-cent India Stock ;

A freehold house, stated to have been valued for probate two years ago at £1,800.

What sums would you advise an intending purchaser to give :

(1) For A's reversion ;

(2) For the interests of A and B together?

*2. A, aged 45, is entitled contingently, on his surviving B, a bachelor, aged 76, and provided the latter leave no issue, to £500 a year for life, secured on an estate of ample value. What sum would you advise a purchaser to give for A's interest, if he wanted to make an investment on the basis of a 4 per-cent yield, free of income tax, throughout?

*3. Find a value for the following reversionary interest and discuss any points which seem to call for special mention.

A, the reversioner, a male aged 45, is entitled if living at the death of B, a female aged 70, to a fund, out of which an annuity of £100 is payable during the life of a lady aged 65.

The fund is at present invested as follows:

£5,000 $2\frac{1}{2}$ per-cent Consols;

£5,000 India $3\frac{1}{2}$ per-cent Stock;

£120 Great Indian Peninsula Railway "A" Annuity, terminable 1948;

£5,000 5 per-cent War Loan, 1929–1947;

£2,000 New Zealand 4 per-cent Inscribed Stock, 1929.

4. Discuss generally the suitability of investments in reversionary interests for (1) the funds of insurance companies and (2) for individuals. Say what class of interests are preferable and why, and specify the particular features in such transactions which would influence your opinion as to their suitability.

What is a base fee and how would you deal with the offer of sale of such a security?

*5. Each of two spinsters, aged 65 and 60, is entitled for life to one-half of the income from

£10,000 $2\frac{1}{2}$ per-cent Consols;

£10,000 London and North-Western Railway 3 per-cent Debenture Stock;

£5,000 New South Wales $3\frac{1}{2}$ per-cent Stock, 1924,

with power to appoint to any husband who may survive her, a life interest to the extent of not more than one-half of her share of income. At the death of the first of the two life tenants to die, the survivor takes for her life the income of the deceased sister also, subject to any life interest appointed in favour of the deceased's husband.

What is the value of the reversion to the whole fund, and what covering policies should be effected?

6. A.B., aged 30, is entitled, if he survives his father, aged 60, to the reversionary life interest in :

£5,000 New Zealand 4 per-cent Stock, 1929 ;

£5,000 Chinese Government 8 per-cent Loan, 1918 (redeemable by 1928, by annual drawings at par).

A.B. desires to sell his interest, but to retain for 5 years the option to repurchase on payment to the purchaser of all sums disbursed by him, accumulated at compound interest. Indicate fully how you would ascertain the value of his interest on this basis.

* "A Short Collection of Actuarial Tables" will be supplied for use in answering these questions.

PROCEEDINGS OF THE INSTITUTE.—Session 1918-1919.

First Ordinary Meeting, 16 *December* 1918.

The President (Mr. Geoffrey Marks, O.B.E.) in the Chair.

The President delivered an Inaugural Address.

Second Ordinary Meeting, 24 *March* 1919.

The President (Mr. Geoffrey Marks, O.B.E.) in the Chair.

A discussion on "Aviation and Life Assurance" was opened by Dr. L. E. Stamm. The following gentlemen took part therein:—Mr. A. D. Besant; Major-General Sir W. S. Brancker, K.C.B., Lt.-Colonel Mervyn O'Gorman, C.B., Brigadier-General J. G. Hearson, D.S.O., Lt.-Colonel Flack (visitors); and the President.

Third Ordinary Meeting, 28 *April* 1919.

The President (Mr. Geoffrey Marks, O.B.E.) in the Chair.

Mr. Andrew Rutherford Davidson, F.F.A., was elected an Associate of the Institute.

A paper entitled "Group Insurance" was read in abstract by the Author, Mr. P. H. McCormack.

The following gentlemen took part in the discussion:—Messrs. F. P. Symmons, W. Schooling (a visitor), Sir Alfred Watson, Messrs. C. W. Kenchington, E. B. Nathan, H. H. Austin, S. G. Warner, and the President.

The Seventy-second Annual General Meeting, 2 *June* 1919.

The President (Mr. Geoffrey Marks, O.B.E.) in the Chair.

The proceedings at the Annual General Meeting will be found on page 396.

REPORT, 1918-1919.

The Council have the pleasure to report to the Members upon the work of the Institute during the Session of 1918-1919, the seventy-first year of its existence.

There has been a *decrease* of 19 in the total number of members, as compared with the previous year. At the end of the official year in which the Institute was incorporated by Royal Charter the number of Members was 434; twenty-three years later, at 31 March 1908, it was 1,009. Since that time the numbers have been as follows:

On 31 March	Fellows	Associates	Students	Corresponding Members	Total
1909	254	325	400	19	998
1910	259	335	348	21	963
1911	267	339	308	20	934
1912	278	354	268	20	920
1913	282	355	252	19	908
1914	295	358	238	19	910
1915	304	361	263	17	945
1916	308	345	247	17	917
1917	303	344	231	18	896
1918	295	332	215	18	860
1919	288	330	205	18	841

The following schedule shows the additions to, and the changes and losses in the membership which have occurred during the year ending 31 March last:

Schedule of Membership, 31 *March* 1919.

	Fellows	Associates	Students	Corres-ponding Members	Total
i. Number of Members in each class on 31 March 1918 .	295	332	215	18	860
ii. Withdrawals by					
(1) Death . . .	7	7	7	...	27
(2) Resignation or otherwise . .	1	2	3	...	
	287	323	205	18	833
iii. Additions to Membership					
(1) By Election . .	...	...	...	...	8
(2) By Examination .	...	...	1	...	
(3) By Re-instatement	1	3	3	...	
	288	326	209	18	841
iv. Transfers					
(1) By Examination: *from* Associates	...	...	...	...	...
to Fellows .	...	...	...	...	...
	288	326	209	18	841
(2) By Examination: *from* Students	...	...	4	...	...
to Associates .	...	4	...	...	...
v. Number of Members in each class on 31 March 1919 .	288	330	205	18	841

[*Continued on page* 392.]

Dr. *Revenue Account for the*

1918. £ s. d.		£ s. d.	1919. £ s. d.
	Amount of Funds at the beginning of the year—		
8,045 3 7	General Fund (including Stock of Publications, other than *Journal*)	8,184 13 2	
433 18 2	Messenger Legacy Fund	446 18 6	
372 16 1	Brown Prize Fund	383 19 9	
793 15 3	G. F. Hardy Memorial Fund	832 19 5	
9,645 13 1			9,848 10 10
	G. F. Hardy Memorial Fund (A. J. Cook's Bequest)		135 11 3
	Subscriptions—		
751 16 0	Fellows	804 6 0	
532 7 0	Associates	565 19 0	
126 0 0	Students	157 10 0	
13 2 6	Probationers	32 11 0	
1,423 5 6		1,560 6 0	
3 3 0	*Fines on Reinstatement*	...	
1,426 8 6		1,560 6 0	
57 15 0	*Less* Waived and returned to Members and Probationers on Naval and Military Service	48 6 0	
1,368 13 6		1,512 0 0	
...	One Annual Subscription Compounded for	10 10 0	
	Entrance Fees—		1,522 10 0
4 4 0	*Associates*	...	
3 3 0	Students	5 15 6	
...	Probationers	18 18 0	
7 7 0			24 13 6
76 19 11	Balance of Publications Account		134 11 0
	Dividends and Interest—		
295 14 4	General Fund	215 19 3	
13 0 4	Messenger Legacy Fund	13 8 2	
11 3 8	Brown Prize Fund	11 10 5	
39 4 2	G. F. Hardy Memorial Fund	19 2 2	
359 2 6			260 0 0
£11,457 16 0			£11,925 16 7

Publications Account for the

£ s. d.		£ s. d.
249 0 2	Stock (excluding *Journal*) at the beginning of the year	323 19 2
105 12 8	*Cost of Reprint of Text-Book, Part II*	...
...	Cost of Reprint of Sir G. F. Hardy's Prize Essay on Friendly Societies	37 0 0
33 3 4	Binding, Advertising, &c.	31 1 10
76 19 11	Balance	134 11 0
£464 16 1		£526 12 0

Balance Sheet,

£ s. d.	LIABILITIES.	£ s. d.	£ s. d.	£ s. d.
8,184 13 2	General Fund		8,242 10 7	
233 9 2	Messenger Legacy Fund	233 9 2		
213 9 4	Accumulated Dividends	226 17 6		
446 18 6			460 6 8	
200 0 0	Brown Prize Fund	200 0 0		
183 19 9	Accumulated Dividends	195 10 2		
383 19 9			395 10 2	
767 15 9	G. F. Hardy Memorial Fund	903 7 0		
65 3 8	Accumulated Dividends	84 5 10		
832 19 5			987 12 10	
				10,086 0 3
...	Reserve for Income Tax on War Loan Stock			145 13 0
20 5 11	Sundry unpaid Accounts			38 8 3
...	Examination Fees, 1919			10 10 0
£9,868 16 9				£10,280 11 6

year ending 31 *March* 1919. Cr

1918. £	s.	d.		£	s.	d.	1919. £	s.	d.
			Journal—						
260	3	4	Printing of Nos. 267, 268	399	10	5			
43	2	6	Editorial Expenses	43	2	6			
303	5	10		442	12	11			
102	5	2	*Less* Sales during the year	128	8	6			
201	0	8					314	4	5
25	8	3	Library—Binding, Purchases, &c.				49	3	2
25	6	4	Meetings				21	12	0
...			Legal Charges				40	16	0
600	0	0	Office Expenditure—Rent	600	0	0			
520	9	0	Salaries	534	6	2			
49	17	2	House expenses	55	5	1			
30	7	2	Fire and other Insurance	32	14	10			
78	17	11	Stationery and Printing	149	7	1			
20	5	0	Postage and Telegrams	38	2	11			
57	13	8	Sundries	4	4	8			
1,357	9	11					1,414	0	9
9,848	10	10	Amount of Funds at the end of the year as per Balance Sheet				10,086	0	3
£11,457	16	0					£11,925	16	7

Examined and found correct, 28 *April* 1919.

W. MOUAT JONES,
E. W. HUMPHRY,
STANLEY HAZELL, } *Auditors.*

year ending 31 *March* 1919.

£	s.	d.		£	s.	d.
140	16	11	Sales (excluding *Journal*)	277	5	8
323	19	2	Stock (excluding *Journal*) at the end of the year	249	6	4
£464	16	1		£526	12	0

Examined and found correct, 28 *April* 1919.

W. MOUAT JONES,
E. W. HUMPHRY,
STANLEY HAZELL. } *Auditors.*

31 *March* 1919.

£	s.	d.	ASSETS.	£	s.	d.
1,875	0	0	£3,000 Natal 3 per-cent Inscribed Stock	1,875	0	0
726	5	0	£1,000 Dominion of Canada 3½ per-cent Registered 1930–50 Stock	726	5	0
700	0	0	£1,000 New South Wales 3½ per-cent Inscribed 1930–50 Stock	700	0	0
357	15	0	£600 Belgian Government 3 per-cent Sterling Loan of 1914	357	15	0
4,275	0	0	£4,500 5 per-cent War Stock, 1929–47.	4,275	0	0
400	0	0	£400 National War Bonds, 4 per-cent, 1927	400	0	0
323	19	2	Stock of Publications (excluding *Journal*) in hand	249	6	4
...			Cash on Deposit Account	250	0	0
355	18	10	Cash on Current Account and in hand	480	18	2
87	3	0	Subscriptions in Arrear	63	0	0
767	15	9	£954. 5*s*. 6*d*. 5 per-cent War Stock, 1929–47 (G. F. Hardy Fund)	903	7	0
£9,868	16	9		£10,280	11	6

The Stock Exchange Securities are taken at the values at which they stood on 31 March 1916, and at cost price in the case of any acquired since that date.

Examined and found correct, 28 *April* 1919.

W. MOUAT JONES,
E. W. HUMPHRY,
STANLEY HAZELL, } *Auditors.*

There are also 169 candidates admitted as Probationers, and 73 as Students conditionally on their passing Part I of the Examination. These are not included in the above Schedule of Membership. The numbers in these two classes since 31 March 1913 have been as follows:

On 31 March	Probationers	Conditional Students	On 31 March	Probationers	Conditional Students
1914	200	67	1917	173	67
1915	188	72	1918	156	70
1916	172	73	1919	169	73

The Council have, with great regret, to report the loss by death, since the last Annual Meeting, of five Fellows, Messrs. Vyvyan Marr, J. G. Priestley, H. T. Kay Robinson, C. Stevens, and H. Archer Thomson; seven Associates, Messrs. W. Borland, E. C. Coote, F. Defries, J. C. Hurley, H. B. Keable, A. G. Portch, and W. G. Titmuss; and five Students, Messrs. G. L. L. Carter, T. Middleton, H. J. Mills, E. A. Newland, and A. R. D. Watson. Eight of these Members, namely, Lieut.-Col. H. T. Kay Robinson, D.S.O. and Bar, Captains F. Defries and T. Middleton, Lieutenants H. J. Mills and E. A. Newland, Lance-Corporal J. C. Hurley, Lieutenant H. B. Keable, R.N.V.R., and Naval Instructor G. L. L. Carter, rendered the supreme sacrifice in the Nation's cause. The Council have in addition to record with much regret the deaths of twenty-one Probationers of the Institute, Captains E. C. K. Clarke, M.C., and J. H. Orr, Lieutenants J. W. E. Alexander, E. M. Dove, M.C., R. G. Gale, A. S. Gregory, M. H. Grigg, F. G. Goodyear, M.C., and D. A. Roberts, Lance-Corporal F. L. Bristow, Gunner C. G. Gifkins, and Privates G. F. T. Ascott, V. J. Austin, W. Bradley, H. W. Brown, R. Cole, T. G. Cunliffe, A. R. Harriss, C. P. Maddox, C. S. Shilson, and A. D. Steed, whose lives were also given in the service of their King and Country. The Council hope at an early date to place in the Hall of the Institute a memorial of the Members and Probationers who have fallen in the War.

By the death of Mr. Vyvyan Marr the profession has lost one of its most loyal and valued Members. Mr. Marr had served on the Council of the Institute, and at the time of his death was Chairman of the Board of Examiners.

Messrs. C. Stevens and J. G. Priestley were known to but few of the present generation of actuaries. They were the last surviving Members of the Actuaries' Club who became Fellows of the Institute under the Charter.

The Annual Subscriptions and the Entrance Fees appearing in the Revenue Account amounted to £1,547. 3*s*. 6*d*., as compared with £1,376. 0*s*. 6*d*. in the previous year. The Income and Expenditure for the year were £1,941. 14*s*. 6*d*. and £1,839. 16*s*. 4*d*. respectively.

At the close of hostilities the total number of Members and Probationers who had been on service with the Army and Navy was 428, of whom 75 had been killed in action or died of wounds.

On behalf of the Institute the Council have submitted to His Majesty The King a Loyal Address on the signing of the Armistice.

The amended Bye-laws regulating the election of Members of the Council, duly passed and confirmed at Special General Meetings of the Fellows and Associates of the Institute, have received the sanction of the Privy Council.

The Council have accepted with much regret the resignation of Mr. J. Spencer as Joint Honorary Librarian. Mr. Spencer held the office for ten years, and the thanks of the profession at large are due to him for the valuable services thus rendered. Mr. A. D. Besant has been appointed to fill the vacancy.

The Council desire to place on record their appreciation of the work of the Board of Examiners in preparing a course of reading for the guidance of students. The course of reading has been published in the *Journal*, and is issued as part of the revised Regulations and Syllabus of Examinations.

As already announced, the Council have decided to resume the Examinations of the Institute. The first to take place under the new Regulations will be held from the 23 to the 25 June next.*

During the Session a special course of Lectures was arranged for Students and Probationers preparing for Parts I and II of the Examinations. The Council take this opportunity of thanking those who have undertaken to deliver the Lectures.

The Council have in contemplation one or more courses of Lectures, to be given at the Institute during next Session, on subjects included in the Syllabus for Parts III and IV of the Examinations as well as on subjects outside the Syllabus but of general interest to the profession.

The Council have been approached by the Royal Patriotic Fund Corporation to undertake the valuation of the Funds under its control. In continuing this work, which was performed by the Institute some years ago, the Council believe that their action will meet with the hearty support of the Members.

The stock in hand of the Institute publications on 31 March was as follows:

No. of Copies	Description of Work
29,716	Parts of *Journal.*
726	Index to Vols. 1 to 40.
1,582	*Text-Book,* Part I (Revised Edition).
375	*Text-Book,* Part II (Second Edition).
630	Government Joint-Life Annuity Tables.
728	Select Life Tables.
14	A Short Collection of Actuarial Tables (New Edition).
844	Frequency-Curves and Correlation, with Addendum and Errata (W. P. Elderton).
38 *in cloth* 2,308 *in paper*	Lectures on Finance and Law (Clare and Wood Hill).
1,525	Lectures on the Companies Acts (A. C. Clauson).
1,187	Lectures on the Law of Mortgage (W. G. Hayter).
693	Lectures on the Measurement of Groups and Series (A. L. Bowley).
1,375	Lectures on the Construction of Tables of Mortality, &c. (Sir G. F. Hardy, K.C.B.).
842	Lectures on Stock Exchange Investments (J. Burn).
1,492	Lectures on Friendly Society Finance (Sir A. W. Watson).
315	South African War Mortality (F. Schooling and E. A. Rusher).
241	Life Assurance Law (A. R. Barrand).
644	British Offices' Valuation Tables.
642	British Offices' $2\frac{3}{4}$ per-cent Temporary Annuity Values.
135	Transactions of the Second International Congress of Actuaries.
781	Index to Transactions of Seven International Actuarial Congresses.
1,500	Examination Questions, 1912–15.

12 *May* 1919.

* For Results see pp. 394–5.

EXAMINATIONS, JUNE 1919.

Examinations were held on the 23rd, 24th and 25th June. 1919, at London, Liverpool, Norwich, Edinburgh, Dublin, Melbourne, Sydney, Wellington, Montreal, Toronto, Ottawa, Winnipeg, Bombay, and Calcutta, with the following results:

Part I.—Section A.

Seventy candidates sent in their names, of whom fifty-nine presented themselves, and twenty-six passed, namely:

Allen, F. D. C.	Ibbotson, L. E.	Stark, A. W.
Chapman, H. V.	Johnston, W. N.	Tharp, H. W.
Cope A.	Knowles, M. B.	Thakur, B. T.
Douglas, F. C. R.	Long, J. A.	Thomas, J. H.
Fassel, E. G.	Milnes, H. L.	Unthank, H. W.
Freeman, H.	Murray, J. R. C.	Walker, J. R.
Goodfellow, P.	Perryman, F. S.	Walker, R. B.
Hallett, H. J.	Reynolds, P. C.	Willows, C. E.
Hooker, P. F.	Southwell, M. G.	

Part I.—Section B.

Forty-five candidates sent in their names, of whom thirty-seven presented themselves, and eighteen passed, namely:

Allen, F. D. C.	Ibbotson, L. E.	Perryman, F. S.
Cope, A.	Johnston, W. N.	Polden, L. S.
Fassel, E. G.	Knowles, M. B.	Thomas, J. H.
Gibberd, J. A.	Long, J. A.	Thakur, B. T.
Goodfellow, P.	Mann, A. H.	Unthank, H. W.
Hooker, P. F.	Murray, J. R. C.	Warren, L. A. H.

Part II.

Thirty-seven candidates sent in their names, of whom thirty-five presented themselves, and sixteen passed, namely:

Capon, G. W.	Innes, F. F.	Rider, W. W. H.
Clarke, J. H.	Knowles, M. B.	Rutherford, C. D.
Coleman, H. D.	Mabon, J. B.	Shrewsbury, A. H.
Fassel, E. G.	McLean, P. S.	Thorpe, A. H.
Hocking, W. S.	Maddex, G. H.	
Houston. C. C.	Marriott, A.	

Part III.—Section A.

Twenty-five candidates sent in their names, of whom twenty-four presented themselves, and sixteen passed, namely:

Barrett, C. C. C.	Denmark, R. J.	Weyer, D.
Blake, W. T. C.	Johnson, A. N.	White, O. D.
Brown, S. P.	Klagge, O. C. J.	Wickens, C. H.
Carpmael, C.	Moore, W. R.	Wilson, A. B.
Chatham, E. F.	Savory, D. S.	
Davidson, A. R.	Watson, A. D.	

Part III.—Section B.

Eighteen candidates sent in their names, all of whom presented themselves, and four passed, namely:

Barrett, C. C. C.	Klagge, O. C. J.
Chatham, E. F.	Savory, D. S.

Part IV.—Section A.

Twenty-seven candidates sent in their names, of whom twenty-six presented themselves, and fifteen passed, namely:

†Brenton, W. P.	†McCormack, P. H.	†Smither, H. B.
†Brown, P. G.	†Menzler, F. A. A.	†Traversi, A. T.
†Evans, A. W.	†Paton, A. G.	†Tyler, V. W.
†Fielder, T. L.	†Robertson, F. W.	†Underwood, R. E.
†Hustwitt, W. E.	†Searle, A. J.	†Wilton, H. G.

Those marked (†) have now completed the Examination for the Class of Fellow.

By Order of the Council,

A. C. THORNE,

Chairman of Board of Examiners.

W. PALIN ELDERTON,
H. M. TROUNCER,

Joint Honorary Secretaries.

PROCEEDINGS AT THE ANNUAL GENERAL MEETING.

The Seventy-Second Annual General Meeting of the Institute of Actuaries was held in Staple Inn Hall, Holborn, on Monday, 2 June 1919, Mr. Geoffrey Marks, O.B.E. (President), in the Chair. Mr. A. Levine (Hon. Secretary) read the Notice convening the Meeting. The Minutes of the preceding General Meeting were read and confirmed.

The President said that the first thing which attracted attention in the Report was the progressive decrease in the number of their members. That could not be attributed to the War, because the decrease began as far back as 1909 and had been going on ever since. When he looked at the figures he asked himself whether it could possibly be the result of the comparatively limited outlook that there was for the profession in the days when the decline in membership first began. By limited outlook he meant in the sense that the number of Actuaries was getting greater than the possible openings for their activities and energy. If that were the real explanation of the decrease in numbers, he hoped that the action which the Council had taken in the last year or so would lead to a comparatively rapid restoration of the numbers to the high-water mark which they reached in 1909.

Naturally, a prominent place in the Report was taken by their record in the country's cause. He thought they might congratulate themselves that the Institute had done its duty nobly. A large proportion of their eligible members had gone on service, and of that number a high proportion had laid down their lives. That, he thought, was a testimonial to their loyalty and bravery, and it would be for the Council to consider, in the course of the coming year, what form of memorial should be placed in the Hall to their memories. He hoped that the memorial which they put up might have some reference, not only to the service which those men had rendered, but to the end which they had attained in that service.

The only deaths to which he would refer were those of Vyvyan Marr and Archer Thomson, both of them loyal servants of the Institute—the former cut off in the height of his usefulness and the latter giving in after a long struggle with indifferent health. Marr was a conscientious man in all the aspects of his life and Archer Thomson had almost a genius for friendship, which endeared him to all those with whom he came in contact.

The consideration of the numbers of those who had lost their lives led him to mention another point, and with some emphasis. It was obvious to him, as well as to the older men amongst them, that the old order with which they were familiar was changing, not only in the Institute, but everywhere outside it. For the older among them, whether they were individuals or institutions, it must be a painful process, and whether or not they could get any good out of it depended on the amount of ease and unselfishness with which they subordinated their private interests to the common good. There was no doubt that many of the changes in the circumstances in which they were living and were going to live were such that old institutions such as their Institute and old individuals like himself must either adapt themselves or perish. The individual perishing did not much matter, but it was not thinkable that an Institute such as that should perish or that its usefulness should be in any way abated, and his conviction was (and he hoped they would share it with him) that out of the turmoil and trouble through which they had passed the Institute might arise stronger and more powerful than it had ever been before, touching more important and wider aspects of life and dealing with them as their training and experience ought to enable them to do. He did not know that they considered often enough that the Institute was the centre of their profession all over the world. The members, as a body, had not so many opportunities of realizing that as had the members of the Council and the President. It was an aspect of the matter, however, which he would ask them to bear in mind, because he could assure them that what they did there was closely scrutinized throughout

the world, and the example which they set would be followed wherever their profession and its tenets had penetrated, and that area was practically the whole world—China and Japan, as well as more advanced countries like America and the Colonies.

In considering to what extent the usefulness of the Institute might be increased, the Council had devised certain methods and plans which were gradually maturing. The modification of the syllabus and the resumption of the examinations in a different form, and at more frequent intervals, was one method which the Council felt would help to make up the ground which had been lost by so many of the younger members during the last four or five years. The Special Lectures which the Council had instituted for the benefit of candidates in Parts I and II had been a great success, and in the name of the Council he would like to thank those gentlemen who gave them. Further courses of a more advanced character, and of a scope somewhat outside the range of their previous interests, were being arranged. The Council had decided to have at least two courses of Lectures, one dealing with economics and banking and one with general statistics, in contradistinction to their vital statistics.

As regarded the first, he was authorized to approach Professor Foxwell, and he was glad to say that he had agreed to give a course of Lectures which would be somewhat more extensive than the Council originally intended, because Professor Foxwell told him that from his experience at the School of Economics, if any really good results were to come out of such a course as they proposed, it must be a somewhat extended course. As at present advised, Professor Foxwell thought that there should be three sessions, in each of which ten Lectures should be delivered. The Lectures would be delivered once a week at some convenient time, and they would be open to all members of the Institute. Neither in that course nor in the statistical course, the details of which were not yet settled, would there be included any examinations, so that would free the minds of a good many of them from a certain amount of anxiety. The Council felt that the members who attended those Lectures, and desired to show to their fellows and to the world that they had attained a certain amount of proficiency in the subjects dealt with, should have an opportunity given them to prove that proficiency, and they therefore had agreed that, in order to encourage original work, Papers should be invited, not necessarily on any specific point dealt with in the Lectures, but on some subject chosen by the student himself. Papers which were considered to be of sufficient merit would be rewarded by prizes out of the funds at the disposal of the Institute, and the Papers would be published in the *Journal* if they were appropriate. Personally, he thought that if they were not suitable for the *Journal*, either owing to their length or some other cause independent of their merit, but, in the opinion of the Council, were of sufficient merit to deserve publication, then they might be published at the expense of the Institute. However, that was only his own personal opinion and, so far, he had not submitted the idea to the Council. He would like to emphasize the fact that the continuance of those opportunities for a higher and wider education must depend very much upon the favour which was accorded them by the members. It was going to be a considerable cost to the Institute to initiate those courses, and the only justification for the expense would be the interest shown and the results secured.

Another indication that the Council was alive to the tendencies of the times was to be found in the new regulations for the election of members of the Council. They had amended their bye-laws on the democratic basis which was now fashionable. He thought the method which the Council had devised tended to stimulate interest in their domestic concerns, and that was all to the advantage of the Institute as a whole. The Council had also introduced regulations as to the nominations for the Presidency, which were as democratic as those which pertained to the election of members of the

Council. Although by their bye-laws they could not go outside the Council to secure nominations for the Presidency, yet the Council had arranged that any member of that body might nominate any other member who was qualified to act as President, quite independently of seniority or any other considerations than those of merit. If more than one nomination were made, the question as to who should occupy the Presidential Chair would be settled by ballot.

He thought he had said enough to show that the governing body was doing its part in the new order of things which they anticipated. The members must help. There should be no body of men engaged in business or professional work better qualified than they were to achieve success, not only in their ordinary work, but in any other vocation of life. He had often thought, and he believed it was true, that accuracy and imagination were the prime factors in a successful career. Since their early mathematical training must tend to accuracy, and seeing that the fact of so large a portion of their training and work being devoted to the application of the theory of probabilities to the affairs of life must stimulate imagination, members of the Institute, if their early training was followed up and the qualities which they had learned were properly developed, ought to secure success in every walk of life to which they chose to turn. He hoped members would bear that in mind and try to realize that the policy of the Council, which he also hoped would be the policy of the Institute, was to encourage a wider and more practical outlook, not only upon the affairs with which they had had to deal in the past, but on affairs which were of vital importance in the life in which they moved.

Mr. R. Todhunter, in seconding the motion, said he believed the report would be regarded as a satisfactory record for a period which, perhaps, of all periods in the history of the Institute, had the best claim to the over-worked title of a transition period. Satisfactory, however, as the report was, it was not an entirely complete record of the activities of the Council or of its executive members; a few unconsidered trifles had been passed over. First and foremost, he would mention the public services rendered by the President as a member of two Royal Commissions—the Income Tax Commission and the Commission on Decimal Coinage. It must, he thought, be as unprecedented in the history of the Institute as it was gratifying that the head of the profession in this country should be serving on two such bodies. Then, again, the Council had taken no credit for the very interesting and instructive meetings which were held in March and April. Those who had had experience in the arranging of meetings would know that their warmest thanks were due to the executive officers for having given them those two gatherings. One other incident worth noting was the remarkable conjuring trick by which the hon. librarians had succeeded in accommodating all the books of the library in the cloak room. In recording the various happenings of the session they must remember that the really great event distinguishing it from every other session in the history of the Institute was the coming back of their members from Service. While they deeply deplored the loss of those who had not come back, they welcomed and congratulated those who were once again with them.

The President mentioned that one of the earliest questions which the Institute would have to tackle when they re-assembled concerned the admission of women. He imagined that, by the time they resumed, most of their members would have returned from the Front, and they would be a full body, able to deal with that question effectively, and he hoped, finally.

Election of Officers.

The report having been unanimously adopted, a ballot was then taken for the election of five members of the Council. The President subsequently

announced that the Officers and Council for the ensuing year would be as follows:—

President.

Geoffrey Marks, O.B.E.

Vice-Presidents.

Arthur Digby Besant, B.A.
Joseph Burn, C.B.E.
Abraham Levine, M.A.
James Douglas Watson.

Council.

Samuel John Henry Wallis Allin, C.B.E.
*Herbert Henry Austin.
*Henry James Baker.
*Louis Ernest Clinton.
Lewis Frederick Hovil.
Charles William Kenchington.
Owen Kentish.
George James Lidstone, F.R.S.E.
*Harold Edward William Lutt.
Reginald George Maudling.
Sir George Ernest May, K.B.E.
Henry Edward Melville.
William Charles Sharman.
Edward Robert Straker.
*Ernest Charles Thomas.
Alfred Charles Thorne.
Edward William Townley.
Samuel George Warner.
Sir Alfred William Watson.
Arthur Thomas Winter.
Ernest Woods.
William Arthur Workman.

Treasurer.

William Peyton Phelps, M.A.

Honorary Secretaries.

William Palin Elderton.
Harold Moltke Trouncer, M.A.

* New Members of Council.

On the motion of Mr. R. L. Elderton, seconded by Mr. H. L. Trachtenberg, Messrs. E. W. Humphry, Stanley Hazell, and D. M. Carment, were elected auditors for the ensuing year.

Mr. A. R. Barrand, M.P., in proposing a vote of thanks to the President, Vice-Presidents, Council and Officers, including Mr. Jarvis, the Assistant-Secretary, for their services during the past year, said that very few words were needed to ensure a unanimous vote. They all knew what the President had done for them during the past year; his work spoke for itself. They congratulated themselves most heartily that, in the difficult and critical times they had passed through, they had had such a President as Mr. Geoffrey Marks. They were also grateful to the Vice-Presidents and the members of the Council for what they had done. As to the Assistant-Secretary, they might regard Mr. Jarvis as the permanent civil servant who did the major part of the work, and he thoroughly deserved their warmest thanks.

Mr. Henry Cockburn seconded the vote, which was carried.

The President, in acknowledging the resolution, remarked that he was particularly pleased to hear the references to Mr. Jarvis, for he had fully deserved them.

A vote of thanks was accorded to the auditors on the proposition of Mr. E. F. Spurgeon, seconded by Mr. H. E. W. Lutt, and was acknowledged by Mr. Hazell.

The proceedings then terminated.

Additions to the Library.

The following works have been added to the Library since the publication of the *Journal* for October 1918:

By whom presented (when not purchased).

Actuarial Society of America.

Transactions, 1918. — *The Society.*

Containing, *inter alia*—

"An Analysis of Claims for Total and Permanent Disability Benefits", by A. Hunter.

"Valuation of Bond Holdings of a Life Insurance Company", by F. H. Johnston.

"Note on Mortality by Habits Representation", by P. H. Evans.

"Note on Mean Population", by J. S. Thompson.

Actuarial Society of America and Association of Life Insurance Medical Directors.

Report of Joint Committee. Standard Mortality Ratios incident to variations in the Height and Weight among Men. New York. 1918. — *The Actuarial Society of America.*

Actuarial Society of Scandinavia.

Transactions, 1918–19. — *The Society.*

American Academy of Political and Social Science (March 1917).

8vo. Philadelphia. — *Purchased.*

Containing, *inter alia*—

"Life Annuities", by M. A. Linton.

"Insurance on sub-standard lives", by A. Hunter.

"The problem of cash surrender-values and cash loans", by J. B. Lunger.

"Mutualization of Life Insurance Companies", by M. M. Dawson.

"Conservation of life by Life Insurance Companies", by Dr. L. K. Frankel.

"Group Insurance", by R. B. Trousdale.

"The exemption of Life Insurance Funds from taxation", by Dr. B. D. Mudgett.

"The Disability Insurance Policy", by A. P. Woodward.

"Methods of insuring Workmen's Compensation", by H. E. Ryan.

"The Calculation of Workmen's Compensation Premium Rates", by C. E. Scattergood.

American-Canadian Mortality Investigation.

Based on the Experience of Life Insurance Companies of the United States and Canada during the years 1900–1915, inclusive of Policies issued from 1843 to 1914, inclusive. La. 8vo. New York. 1918. — *The Actuarial Society of America.*

American Mathematical Society.

Transactions, 1918–19. — *The Society.*

By whom presented (when not purchased).

American Statistical Association.

Transactions, 1918–19. — *The Association.*

***Anson (Sir W. R., Bart.).**

Principles of the English Law of Contract and of Agency in its relation to Contract. 14th edit. 8vo. 1917. — *Purchased.*

***Ashburner (—).**

Concise Treatise on Mortgages, Pledges and Liens, by W. F. Webster. 2nd edit. 8vo. 1911. — *Purchased.*

Association des Actuaires Belges.

Bulletin, 1914. — *The Association.*

Austria.

Bericht der Arbeiter-Unfall-Versicherungs-anstalt für das Königreich Böhmen. 1916. — *Austrian Government.*

Belgium.

La Caisse Générale d'Epargne et de Retraite pendant la Guerre, 1914–18. 8vo. Brussels. 1919. — *Fl. Hankar.*

"Biometrika."

Vol. XII, Parts I and II. — *Purchased.*

Containing, *inter alia*—

"On the Standard Deviation of Adjusted and Interpolated Values of an observed Polynomial Function and its Constants and the guidance they give towards a proper choice of the Distribution of Observations", by K. Smith.

"On the Product-Moments of Various Orders of the normal correlation surface of two Variates", by K. Pearson and A. W. Young.

"The Correlation Coefficient of a Polychoric Table", by A. Ritchie-Scott.

"On a Formula for the Product-Moment Coefficient of any order of a normal Frequency Distribution in any number of Variables", by L. Isserlis.

"On the Mathematical Expectation of the Moments of Frequency Distributions," by A. A. Tchouproff. Part I.

Bowley (A. L.).

The Division of the Product of Industry. An Analysis of National Income before the War. 8vo. 1919. — *Purchased.*

Brockbank (Dr. E. M.).

Life Insurance and General Practice. 8vo. 1908. — *Purchased.*

Carnegie Foundation for the Advancement of Teaching.

Rules for the Admission of Institutions and for the granting of retiring allowances. 4to. New York. 1918.

Twelfth Annual Report of the President and Treasurer. 4to. New York. 1917. — *W. J. H. Whittall.*

Castelnuovo (G.).

Calcolo delle Probabilità. La. 8vo. Milan. 1919. — *The Author.*

	By whom presented (when not purchased).
Casualty Actuarial and Statistical Society of America.	
Proceedings, 1918.	*The Society.*
Containing, *inter alia*—	
"Economic Problems of the World War", by J. D. Craig.	
"The relation between the Actuary and the Statistician", by J. D. Craig.	
"Mortality from external causes among Industrial Policyholders in the Metropolitan Life Insurance Co.", by Dr. L. I. Dublin.	
"Essentials of Family Statistics", by E. W. Kopf.	
Chartered Insurance Institute, Journal of the	
Vol. XXI. 8vo. 1918.	*The Institute.*
Dougharty (H.).	
Notes on Deposit Pension Schemes. 1919.	*The Author.*
Dublin (Dr. L. I.).	
Mortality Statistics of insured wage-earners and their families. Experience of the Metropolitan Life Insurance Co. (Industrial Dept.), 1911–1916 in the United States and Canada. 8vo. New York. 1919.	*The Metropolitan Life Insurance Co.*
Duguid (C.).	
The Stock Exchange. 3rd edit. 8vo. 1913.	*Purchased.*
Economic Society (Royal).	
Journal of the, 1918–19.	*Purchased.*
Frost (P.).	
An Elementary Treatise on Curve Tracing. 4th edit. 8vo. 1918.	*Purchased.*
Galton (Sir Francis), F.R.S.	
Hereditary Genius. An enquiry into its laws and consequences. 8vo. 1914.	*W. Palin Elderton.*
Memories of my Life. 3rd edit. 8vo. 1909. Record of Family Faculties. 4to. 1884.	*Purchased.*
Greenwood (Dr. M.), Jr., and Frances Wood.	
On changes in the recorded mortality from Cancer and their possible interpretation. 1914. The Relation between the Cancer and Diabetes Death-rates. 1914.	*Dr. Greenwood.*
Greenwood (Dr. M.), Jr., and G. Udny Yule.	
The Statistics of Anti-typhoid and Anti-cholera inoculations, and the interpretation of such Statistics in general. 1915.	*Dr. Greenwood.*
***Halsbury (Rt. Hon. Earl of).**	
The Laws of England. Being a complete Statement of the Law of England. With Index and Supplement. 32 vols. 8vo. 1907–1918.	*Purchased.*
***Hanson (A.).**	
The Acts relating to Estate, Increment Value (on Death), Legacy, Succession, and Probate Duties. With Supplement. 6th edit. 8vo. 1911–15.	*Purchased.*

By whom presented (when not purchased).

***Hart (H. L.).**

The Law of Banking, with an Appendix on the Law of Stock Exchange transactions. 3rd edit. 8vo. 1914. — *Purchased.*

Hawtrey (R. G.).

Currency and Credit. 8vo. 1919. — *Purchased.*

Henderson (R.), and Others.

Graduation of Mortality and other tables. 8vo. New York. 1919. — *The Actuarial Society of America.*

Hoffman (Dr. F. L.).

Pauper Burials and the Interment of the Dead in large cities. 8vo. Atlantic City, N.J. 1919. — *The Author.*

Holland.

Archief voor de Verzekerings-Wetenschap. 1918–19. — *The Society.*

Hunter (A.).

An Analysis of Claims for Total and Permanent Disability Benefits. New York. 1918.

Gastric and Duodenal Ulcers. Mortality after operation. New York. 1919. — *The Author.*

***Indermaur and Thwaites.**

Principles and Practice in matters of, and appertaining to, Conveyancing. By C. Thwaites. 3rd edit. 8vo. 1910. — *Purchased.*

Insolera (F.).

Sulla misura dell' aumento di mortalità per effetto indiretto della guerra. Rome. 1919. — *The Author.*

Institute of Bankers.

Journal of the, 1918–19. — *The Institute.*

Insurance Institute of America.

Proceedings of the Tenth Conference. 8vo. 1918. — *The Institute.*

Insurance Institute of New South Wales.

Transactions, 1918. — *The Institute.*

Insurance Institute of Toronto.

Proceedings, 1918–19. — *The Institute.*

Containing, *inter alia*—

"Present-day tendencies in Insurance", by G. D. Finlayson.

"Group Insurance", by W. J. Graham.

"Methods of Valuation for Life Insurance Securities", by D. E. Kilgour.

Isserlis (L.).

The application of Solid Hypergeometrical series to Frequency Distributions. 1914.

The variation of the multiple Correlation Coefficient in samples drawn from an Infinite Population with normal Distributions. 1917.

On the value of a man as calculated from a sample. 1918. — *The Author.*

	By whom presented (when not purchased).
***Jelf (A. E.).**	
Where to find your Law. Being a discursive bibliographical essay upon the various divisions and sub-divisions of the Law of England and the Statutes, Reports of Cases, and Text-Books containing such Law. 3rd edit. 8vo. 1907.	*Purchased.*
Joffe (S. A.).	
Calculation of eighteen more, fifty in all, Eulerian numbers from Central Differences of zero. 1919.	*The Author.*
King Edward's Hospital Fund for London.	
Report of a Sub-Committee of the Executive Committee on Pensions for Hospital Officers and Staffs. Fol. 1919.	*W. J. H. Whittall.*
Laughlin (Dr. J. L.).	
Credit of the Nations. A study of the European War. 8vo. 1919.	*Purchased.*
Life Offices' Association.	
Minutes, etc., of the Meetings of the Standing Committee, 1918–19.	*The Association.*
London Mathematical Society.	*The Society.*
Proceedings, 1918–19.	
Marshall (A.).	
Industry and Trade. 8vo. 1919.	*Purchased.*
Moir (H.), and Others.	
Sources and Characteristics of the principal Mortality Tables. 8vo. New York. 1919.	*The Actuarial Society of America.*
National Health Insurance.	
Reports of Medical Research Committee. 8vo. 1918–19.	*Purchased.*
Parliamentary Papers.	
Acts of Parliament. Income Tax, 1918.	*Purchased.*
Assurance Companies. Returns to the Board of Trade. 1918.	*The Board of Trade.*
Colonies.	
Canada.	
Report of the Superintendent of Insurance for the year 1917. Insurance Companies. Abstract of Statements for the year 1918.	*The Government Insurance Dept.*
New South Wales.	
Friendly Societies, &c. Report of the Registrar for 1918. Official Year Book, 1918. Statistical Register for 1917–18 and previous years. Vital Statistics. Report for 1918 and previous years.	*The Government of N.S.W.*

	By whom presented (when not purchased).
Parliamentary Papers—*continued.*	
Colonies—*continued.*	
New Zealand.	
Government Insurance Department. Triennial Investigation, 1917. Annual Report, 1917. Accident Branch. Report, 1917. Census, 1916, Parts II and IV. Fol. Wellington. 1919.	*The Government of N.Z.*
Victoria.	
Friendly Societies. Fortieth Annual Report of the Government Statist, 1917. Twenty-eighth Annual Report of the Registrar, 1917.	*The Government of Victoria.*
Western Australia.	
Friendly Societies. Report of Proceedings by the Registrar for the year ended 30 June 1918. Statistical Register, 1917–18.	*The Government of W.A.*
Currency and Foreign Exchanges.	
First Interim Report of Committee on Currency and Foreign Exchanges after the War. 1918.	*Purchased.*
Registrar-General. England.	
Eightieth Annual Report of Births, Deaths, and Marriages, 1917. Supplement to the 75th Annual Report of Births, Deaths, and Marriages. Part III. Registration Summary Tables (1901–1910). Fol. 1919.	*The Registrar-General.*
India.	
Life Assurance Companies. Returns of Companies doing business in British India, 1917. Fol. Simla. 1918.	*H. G. W. Meikle.*
Periodicals.	
Accountants' Magazine. American Bankers' Magazine.	*Purchased.*
American Economic Review.	*The Editor.*
Bankers' Magazine.	*Purchased.*
Commercial and Financial Chronicle. 1895–1913. 38 vols. Fol. New York.	*O. T. Falk.*
Economist.	*Purchased.*
Insurance Record. Journal of Political Economy. (Chicago). Post Magazine. Post Magazine Almanack.	*The Editor.*
The Secretary.	*Institute of Secretaries.*
***Ringwood (R.).**	
The Principles of the Law of Bankruptcy. Embodying the Bankruptcy Act, 1914. 12th edit. 8vo. 1915.	*Purchased.*
Royal Astronomical Society.	
Memoirs of the. Vol. LXIII, Part III. 4to. 1918.	*The Society.*
Statistical Society (Royal).	
Journal of the, 1918–19.	*The Society.*

	By whom presented (when not purchased).
Sweden.	
Enskilda Försäkringsanstalter ar 1917, av Kungl. Försäkringsinspektionen.	*The Swedish Government.*
"Thule" Lifförsäkrings-aktiebolaget. Fyrtio ärs verksamhet, 1873–1912. Also Souvenir Volume. 4to. Stockholm. 1917–18.	*The Company.*
Switzerland.	
Rapport du Brueau Fédéral des Assurances sur les Entreprises privées en matière d'Assurances en Suisse. 1916.	*The Swiss Government.*
Teachers' Insurance and Annuity Association of America.	
Handbook of Life Insurance and Annuity Policies for Teachers. Sm. 8vo. New York. 1918.	*W. J. H. Whittall.*
***Underhill (A.).**	
The Law relating to Trusts and Trustees. 7th edit. 8vo. 1912.	*Purchased.*
United States of America.	
Official Publications.	
Connecticut. Fifty-fourth Annual Report of the Insurance Commissioner, 1919.	*The Commissioner.*
Massachusetts. Sixty-third Annual Report of the Insurance Commissioner (Life and Miscellaneous), 1917.	*The Commissioner.*
Wisconsin. Report of the Special Committee on Social Insurance. Madison, Wis. 1919.	*Dr. F. L. Hoffman.*
Vega (G. F. von).	
Logarithmisch-trigonometrische Tafeln. 8vo. Leipzig. 1914.	*Mrs. E. E. Kimber.*
***Walker (Hon. W. G.) and E. J. Elgood.**	
A Compendium of the Law relating to Executors and Administrators. 4th edit. 8vo. 1905.	*Purchased.*
***Wharton (—).**	
Law Lexicon. Forming an epitome of the Laws of England under Statute and Case Law. By E. A. Wurtzburg. 12th edit. 8vo. 1916.	*Purchased.*
Williams (Caroline E.)	
A Welsh Family. 8vo. 1893.	*Mrs. R. H. T. Williams.*
***Williams (Joshua).**	
Principles of the Law of Real Property. 22nd edit. 8vo. 1914.	*Purchased.*
Withers (Hartley).	
War-time financial problems. 8vo. 1919.	*Purchased.*

By whom presented (when not purchased).

Yule (G. Udny).

An introduction to the Theory of Statistics. 5th edit. 8vo. 1919. — *Purchased.*

* For *Reference* only.

Additional copies of works already in the Library:

Farr (Dr. W.).

Vital Statistics. 8vo. 1885. — *Mrs. Vyvyan Marr.*

Hammond (H. P.).

Life Insurance in Groups, 1912–17. St. Paul, Min. 1917. — *The Author.*

Karup (J.).

Reform des Rechnungswesens der Gothaer Lebens-versicherungsbank. La. 8vo. Jena. 1903. — *Mrs. Vyvyan Marr.*

Todhunter (I.).

A History of the Mathematical Theory of Probability. 8vo. 1865. — *Mrs. Vyvyan Marr.*

THE INSTITUTE OF ACTUARIES.

ENCOURAGEMENT OF ACTUARIAL RESEARCH.

The Council of the Institute has resolved that prizes may in future be given out of the Prize Funds for any work of exceptional merit. To help those who may wish to undertake original work, the appended list of subjects requiring investigation has been prepared.

Papers submitted should be concise; many of the subjects indicated are capable of sub-division and a part only of any one chosen may be discussed.

Prizes may also be given for original work on other subjects. To prevent a number of persons working at the same subject, or perhaps in suitable circumstances to arrange for co-operation, it would be well for those who contemplate undertaking work on the subjects indicated to communicate privately with one of the Honorary Secretaries.

List of Subjects.

Mathematical.

Rigid proof of circumstances in which the method of *J.I.A.*, vol. xliv, pp. 293, &c., holds.

Relative accuracy of various two-variables interpolation methods.

Mortality and Other Tables.

How is selection modified by the withdrawal of healthy lives and other influences ?

A discussion of the various aggregate tables that can be evolved by various aggregations of select rates of mortality and the uses to which such work could be put.

The effect of catastrophes on mortality, sickness, &c., and their treatment in constructing mortality and other tables (*e.g.*, War, Plague in India, &c.).

Sickness and mortality rates according to disease.

Social.

The effect of occupation on mortality, sickness, birth rate, &c.

Studies in population density, emigration, &c., in various countries.

Fertility in relation to occupation, locality, duration of marriage, ages at marriage, &c.

Infantile mortality in relation to birth rate, housing, wages, order of birth, &c.

Effect of heredity on mortality and disease.

Wages and cost of living compared with size of family, &c.

Taxable capacity of a country and its connection with social schemes and present liabilities.

Insurance Valuations, Practice, &c.

If selection be exaggerated by a table of mortality, what errors will be introduced into the premiums charged or into a valuation, (*a*) by the select, (*b*) by the full aggregate table ?

An assurance company charges premiums to give a certain form of bonus and its experience is exactly as anticipated. What surpluses would be revealed by valuations assuming various mortality tables based on that assumed in the premiums (*e.g.*, full aggregate table, ultimate table, full aggregate made up from the select table in different proportions) ?

General basis of non-profit rates in view of growing expenditure tax, &c.

Statistics and treatment of under-average lives.

The effect on surpluses of continuous fall (or rise) in interest or of gradual changes in expenses, or mortality or other factors.

The effect on successive surpluses of a life assurance or other fund of unduly stringent valuations.

Pension Funds.

A discussion of cases in which mortality may be heavy, then decrease and become normal, and its effect on pension funds, &c.

The effect of changes in salary scale on liabilities of pension funds.

The dangers of using aggregate tables based on past experience for valuing pension funds, &c. (*i.e.*, aggregate tables of rates of lapse, retirement, mortality, &c.).

Size of family and ages of children, &c., in relation to orphans' pensions.

The application of various money-purchase systems to the provision of pension funds.

Tabulation.

What is the best system of tabulating functions ?

The tables based on British Offices mortality experience omit many useful rates of interest and helpful functions. Are individual ages necessarily the best division ? Special points arise in connection with varying insurances and benefits dependent upon a number of lives (*e.g.*, last survivor annuities for equal ages). The object is to ascertain the most useful tables within certain defined limits.

Financial.

Statistical studies on the relationship between changes in currency conditions and the rate of interest.

Statistical studies of the changes in the relations between prices of securities of different types and in different markets.

Graduation.

The tests of a good graduation. Can a mathematical test be given to show that a suggested graduation is so far from the facts as to be " impossible " ?

Studies in graduation by new methods or variations of old methods—specially as regards select mortality, sickness, &c.

Examples of types of material which particular graduation methods fail to meet.

The relative merits either from a practical or theoretical standpoint of various systems of frequency curves.

General.

New approximate methods of valuation, calculation, &c.

LECTURES ON FINANCE.

The following is the Syllabus of the Lectures to be delivered by Prof. Foxwell at Staple Inn Hall during the current session:

Markets, Speculation and the Function of the Dealer: an omitted chapter in the text-books of economics. Nature and definition of a Market.

Conditions of the efficiency of a great speculative market. Standardisation. Warrants. Dealing in Futures.

Nature and general economic utility of speculative dealing: how distinguished from gambling. Abuses of speculation. Publicity as a remedy.

Recent attacks on the speculative markets. The German Bourse Law of 1896. Legislation against Time dealings.

The Stock Exchange the greatest of all markets: except that the machinery of standardisation, not required, is absent. An essential part of the banking system and money market. Constitution of the London Market. Provincial exchanges.

Methods of Dealing: the Jobber and the Broker: difficulties in regard to the demarcation of their functions. Publicity: marking transactions. Fortnightly Settlement. Financing by contango on the market and by Banks outside. Moneybrokers. Floaters. The outside broker. Comparisons throughout with Paris Bourse and Wall Street.

Securities and their Manufacture. The Issue Houses. The Company Promoter. Trust Companies. The Provincial Stockbroker. Recent intervention of Clearing Banks. Promotion in other countries. Underwriting and the machinery of issue. Cost of Issue. Firm holding, and its effect on values. Giving a quotation. Dealings before allotment.

Classification of Securities. Bonds and Industrials. Trustee Securities *v.* Speculative Counters. Debentures and Preferred Stocks, &c. Bearer Bonds *v.* Registered. Terminable *v.* Non-Terminable Loans. Saleability on Foreign Markets. Accrued Interest. New York parity, &c.

General causes that determine the rise and fall in the values of Securities. Habits of Saving and Consumption. Field of Employment. Effect of movements in the general level of prices. Effects of Money Market and foreign exchange conditions.

Other considerations relating to the problems of investment. Effect of Cyclic Variations. Effect of War conditions. Effect of Market movements. Relative stability of securities. "Floaters" sensitive. Policy of "Nursing" loans.

Theoretical presumptions less valuable than they used to be: usually over-discounted. Intrinsic *v.* Market values. The Market in some respects less efficient than it was: probable reasons. Is Government control as exercised in some foreign countries desirable here? The market has shown itself at least as patriotic and well-informed as the Government. Publicity the simplest and most effective form of control.

BOOKS.—*The most essential are starred. Others will be referred to in the course of the Lectures.*

*Prof. H. C. EMERY.—"Speculation on the Stock and Produce Exchanges of the U.S." *New York.* (1896).

ANDRÉ E. SAYOUS.—"Étude économique et juridique sur les Bourses Allemandes de Valeurs et de Commerce." *Paris* and *Berlin.* (1898).

Prof. H. C. EMERY.—"Futures in the Grain Market." *Economic Journal.* (March, 1899).

Prof. H. C. EMERY.—"Ten years' Regulation of the Stock Exchange in Germany." *Yale Review.* (May, 1908).

*CHARLES DUGUID.—"The Stock Exchange." *Methuen.*

HARTLEY WITHERS.—"Stocks and Shares." *Smith, Elder.* (1910).

CHARLES DUGUID.—"How to read the Money Article." *Effingham Wilson.*

W. C. VAN ANTWERP.—"New York: the Stock Exchange from Within." *Effingham Wilson.* (1913).

*JOSEPH BURN.—"Stock Exchange Investments." *Institute of Actuaries.* (1909).

E. S. MEADE.—"Trust Finance." *New York. Appleton.*

W. J. GREENWOOD.—"Foreign Stock Exchange Practice and Company Laws." 36, *Camomile Street, E.C.*3. (1911).

Obituary.

GEOFFREY YATES HEALD, Probationer of the Institute, Captain, 15th Battalion, Lancashire Fusiliers.

Killed in Action 1 *July* 1916.

THE NATIONALITY OF TETENS.

We regret that, owing to an oversight, for which Dr. Steffensen was in no way responsible, the birthplace of Tetens was stated on p. 309 to be Kiel. From the late Mr. Frederick Hendriks's article in *J.I.A.*, vol. i, p. 14, it appears that he was born at Tetensbüll, Eiderstädt, in the Duchy of Schleswig.—Eds.

ERRATUM.

J.I.A., vol. li, p. 134, line 30, $i+d$ should be $\frac{i+d}{2}$.

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"Actuarial Studies: No. 1, Sources and Characteristics of the principal Mortality Tables. No. 4, Graduation of Mortality and other Tables." Review, 367.

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